

Targeted Search with Horizontal Differentiation in the Marriage Market

Yujing Xu* and Huanxing Yang[†]

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Abstract

We develop a search/matching model in the marriage market with heterogeneous men (a continuum of types) and heterogeneous women (a finite number of types). The model has two distinguishing features. First, men and women are also horizontally differentiated. Second, the search is targeted: each type of woman constitutes a distinctive submarket, and men are able to choose beforehand in which submarkets to participate, but the search is random within each submarket. We show that there is always a unique equilibrium in which men are endogenously segmented into different submarkets, and that the equilibrium matching pattern is weakly positive assortative. We then explore how the equilibrium marriage pattern changes horizontally and vertically as some exogenous shocks occur. In particular, we show that an Internet-induced increase in search efficiency would make the marriage pattern overall more assortative, while an increase in the dispersion of the horizontal match fitness could make the marriage pattern overall less assortative.

JEL Classifications: C78; D83; J12

Key Words: Horizontal Differentiation; Marriage Market; Matching; Targeted Search

1 Introduction

Search/matching is a very useful framework to study the marriage pattern in the marriage market (Burdett and Coles, 1997; Smith, 2006). In these models, men and women are heterogeneous in their vertical types (defined by income and appearance, etc.), and they study the equilibrium matching pattern in the vertical dimension. However, one important feature has not been captured by these models: in the real world, the utility enjoyed by a married couple depends not only on their “vertical” types, but also on the compatibility of their personalities. That is, horizontal differentiation is also an important aspect in the marriage

*Faculty of Business and Economics, University of Hong Kong. Email: yujingxu@hku.hk

[†]Department of Economics, Ohio State University. Email: yang.1041@osu.edu

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market.¹ More specifically, men and women are also horizontally differentiated in terms of tastes/interests/hobbies/characters, and people also care about those attributes when they search for partners. While it is documented that both vertical and horizontal differentiations are important in shaping the sorting pattern in the marriage market,² the existing literature lacks tractable theoretical models that incorporate both dimensions.

Another feature of Burdett and Coles (1997) and Smith (2006) is that they both assume random search. In the real world, however, people usually are able to narrow down their search to potential partners with a certain income or education level. That is, people can usually target their search in the dating market. This targeting ability has been greatly enhanced by the Internet. For instance, on Match.com one can sort people according to their income.

To capture the above-mentioned features in the marriage market, this paper develops a search/matching model with (vertically differentiated) heterogeneous men and heterogeneous women, with the following two novel features. First, in addition to vertical differentiation, there is horizontal differentiation among men and women, which is unrelated to men’s or women’s vertical types. This could reflect the similarity or compatibility between a woman’s and a man’s personalities/interests.³ We model it as an i.i.d. match value between any woman-man pair. Second, search is targeted and one-sided (only men search for women).⁴ In particular, each type of woman (the number of types is finite) constitutes a distinctive submarket. Search is targeted, in the sense that men, whose types are continuous, are able to choose in which submarkets to participate (or which types of women to target) beforehand. Within each submarket, however, search is random. A detailed discussion of the modeling assumptions can be found in Section 3.3.

¹Here the terminologies of vertical differentiation and horizontal differentiation are adopted from the IO literature. Specifically, for vertical attributes everyone agrees on the rankings, but for horizontal attributes different agents might have different rankings. In other literature, vertical attributes are sometimes called “objective” attributes or “common shock,” while horizontal attributes are called “subjective” attributes or “idiosyncratic shock.”

²For instance, Hitsch et al. (2010a) and Herrenbrueck et al. (2016) found that both horizontal and vertical components of preferences are important in driving observed sorting patterns in the marriage market.

³According to Finkel et al. (2012) and the references therein, the matching recommendations at some online dating websites (PefectMatch, PlentyOffish, OkCupid) are based on both similarity and complementarity. In particular, Pepper Schwartz (2006), the academic relationship expert at PerfectMatch, stated that “couples will be more compatible when they are similar on romance impulsivity, personal energy, outlook and predictability, and when they are different on flexibility, decision-making style, emotionality, and self-nurturing.” However, empirical evidence regarding whether greater similarity on attitudes and values or higher compatibility between partners benefits relationships remains weak and inconsistent. To summarize, horizontal match value depends on multi-dimensional traits, and it is still quite random even after controlling for the similarity/complementarity of traits.

⁴More detailed discussion about the assumption of female’s passive role in the marriage market can be found in Section 3.3. The following quote (Long, 2010) from a 43-year old female online dater exemplifies women’s preference for the passive role: “I don’t browse profiles. . . . I sit around and wait, if somebody winks at me, I look and see if I’m interested. If I’m intrigued at all by their profile, I’ll wink back and we’ll start to chat.”

Within our framework, we intend to answer the following research questions. First, how do sorting in the vertical dimension and sorting in the horizontal dimension interact with each other?⁵ Second, how are the matching outcomes in the marriage market affected by the Internet, which makes it easier to find a potential partner? And how is the marriage pattern affected when the horizontal match value becomes more dispersed, which seems to be a trend in modern and post-modern society? In particular, will these changes make the marriage pattern overall more assortative or less assortative in the vertical dimension and in the horizontal dimension? How will these changes affect the utilities and the marrying speeds of different types of men and women? Will they increase or decrease men's or women's inequality in the marriage market? These are important questions, as they are not only important in their own right, but they also have implications for intergenerational mobility.

In our model, the man and the woman in a marriage always get the same utility, which is multiplicative in the man's type, the woman's type, and their horizontal match value. Upon meeting, a man and a woman observe each other's vertical types as well as the horizontal match value. Then they simultaneously decide whether to marry. A marriage is consummated if and only if both agree to marry. Once married, the couple exit the marriage market forever and are instantaneously replaced by two agents of the same vertical types. Another difference from the search/matching models of Burdett and Coles (1997) and Smith (2006) is that, while in their models the meeting rate of all types of men and all types of women are always the same, in our model the meeting rate could be different across types. In particular, we assume an urn ball search/matching technology (in continuous time). As a result, a man, who searches actively, has a constant meeting rate regardless of his type and the submarkets in which he participates; a woman, who does not search actively, has a meeting rate increasing in the men to women ratio in her submarket.

We show that there is always a unique market equilibrium. Moreover, in equilibrium men are endogenously segmented into different submarkets and the matching pattern is weakly positive assortative, with higher types of men searching for weakly higher types of women. In the horizontal dimension, every man and woman has a threshold of match value for each type of partner above which he or she is willing to accept the marriage (the acceptance cutoff). Within each submarket, higher type men, for whom women have a lower acceptance cutoff, are faster to marry (have higher probabilities of being accepted by women), since, all other things being equal, they are more desirable to women. This is also why a marginal type man can be indifferent between two adjacent submarkets. Being the highest type in one submarket and the lowest in the adjacent higher submarket, he trades off a better mate's type with a faster marrying speed. In equilibrium, all submarkets are indirectly linked through the indifference

⁵Throughout this paper, we say that matching is more assortative horizontally if the newlyweds are more compatible horizontally, i.e., the horizontal match values among the newlyweds are higher.

conditions of the marginal types of men. An interesting implication is that in equilibrium changes affecting lower types would have an impact on higher types, a feature different from Burdett and Coles (1997).⁶ In addition, the equilibrium men to women ratio is, in general, higher in a higher submarket, which enlarges the difference between the marrying speeds of the marginal types in two adjacent submarkets and helps sustain the indifference conditions.

The key driving force of the model is the indirect externalities men within the same submarket imposed on each other. First, more men being active in a submarket increases the men to women ratio and hence women's meeting rate in that submarket, which increases women's expected payoff. This makes them more choosy about men (they only accept men with a higher horizontal match fitness) and reduces the attractiveness of this submarket to men. Therefore, although men's meeting rate is constant regardless of the men/women ratio, there still exists an indirect congestion effect for men. The above mechanism also demonstrates the interactions between vertical sorting and horizontal sorting. A change in the vertical sorting pattern alters women's payoffs, which affects their acceptance payoffs or the pattern of horizontal sorting. In the other direction, changes in women's acceptance cutoffs affect men's payoffs in different submarkets, and hence influence sorting in the vertical dimension.

We then conduct comparative statics. First, we investigate the impact of the Internet. Specifically, the widespread use of the Internet in the last two decades reduces agents' search costs and makes meeting with potential spouses easier. In terms of modeling, it means that the Internet increases all men's contact rate. Under some fairly general conditions, we show that this results in an increase in the men's marginal types, which implies a more assortative marriage pattern in the vertical dimension: fewer men are active in higher submarkets. In the horizontal dimension, the equilibrium acceptance cutoffs all increase, which implies a more assortative marriage pattern in the horizontal dimension as well. These predictions are consistent with the empirical evidence documented by Lee (2009, 2016): as online dating websites become more popular, people in Korea become more likely to marry a spouse with the same marital history and similar education levels.

In terms of payoffs, all women are always better off. However, not all men are better off. Specifically, in each submarket the highest and the lowest types of men are in general better off, but the intermediate types might be worse off. Our simulation also indicates the following quantitative results regarding inequalities in the marriage market. First, among the higher submarkets, higher types of women benefit more than the lower types from an increase in the contact rate, while among the lower submarkets the lower types of women benefit more. Thus an Internet-induced increase in search efficiency increases women's inequality (in the marriage market) in the upper tail, but reduces women's inequality in the lower tail. Second, although

⁶This point is elaborated in Subsection 3.3.

within each submarket men's gains may not be monotonic in types, across the submarkets higher types of men gain more than lower types from an increase in the contact rate. Thus, an Internet-induced increase in search efficiency in general increases men's inequality in the marriage market.

Another impact of the Internet is that it makes horizontal targeting easier. That is, agents that share similar horizontal interests/hobbies/traits can now organize horizontal clubs online. We show that by reducing the possibility of getting a low horizontal match value, having horizontal clubs is essentially equivalent to an increase in the contact rate in the baseline model. Therefore, an Internet-induced horizontal targeting has similar qualitative impacts on the marriage market as an Internet-induced increase in search efficiency.

Second, we examine how a change in the women to men ratio affects equilibrium outcomes. When the women to men ratio increases, matching in the vertical dimension becomes less assortative: more men marry higher classes of women and fewer men marry lower classes of women. This prediction is consistent with the change in the marriage pattern in France after WWI (Abramitzky et al., 2011). We then examine the impact of a change in women's type distribution. When the measure of type n women increases, in all the submarkets lower than n the marginal types of men decrease, while in the higher submarkets the marginal types of men weakly increase. In all the lower submarkets, women are worse off, while men are better off. However, in the higher submarkets, either women are worse off and men are better off, or both women and men are unaffected by the change. This shows that shock in a single submarket transmits through the endogenous adjustment in segmentation to the whole market and such transmissions are asymmetric. In the upward direction, the shock transmission may stop at any submarket, while in the downward direction the shock transmission will go all the way to the lowest submarket.

Finally, we study the impact of an increase in the dispersion of the horizontal match fitness, which can be caused by an increase in the variety of personalities or hobbies. As the dispersion of the horizontal match fitness increases, under some fairly general conditions the equilibrium marginal types of men are very likely to decrease. This leads to overall less assortative matching, opposite to the impact of the Internet. The highest type women benefit from a more dispersed match value, but the lower types of women might be worse off. Among men within a submarket, again both the highest types and the lowest types are better off, but the middle types might be worse off. Our simulation results indicate that a more dispersed match value increases women's inequality as well as men's inequality in the marriage market. Moreover, the increase in women's inequality is more significant than the increase in men's inequality, as the endogenous adjustment in men's segmentation (more men in higher submarkets) amplifies women's inequality but dampens men's across submarkets.

Related Literature In economics, the seminal work of Becker (1973) first models marriage formation as a matching process. In a frictionless environment, he studies the marriage pattern with heterogeneous men and heterogeneous women (different vertical types). His main result is that equilibrium matching will exhibit perfect positive assortative matching if the output function of marriage is supermodular in man’s and woman’s types. Later works introduce search friction into Becker’s matching model,⁷ an approach followed by the current paper as well.

The most closely related papers to ours are Burdett and Coles (1997, BC hereafter), Smith (2006), and Jacquet and Tan (2007, JT hereafter), all of which are search and matching models in the marriage market with vertically differentiated men and women. A common difference between our model and theirs is that in their models there is no horizontal differentiation, which plays an important role in our model. With the presence of horizontal differentiation, instead of being accepted for sure or being rejected for sure, the same type of men will be accepted by different types of women with different but positive probabilities. In some sense this acts as a shadow price, although in our marriage model utilities are non-transferrable.

In a completely random search model, BC show that equilibrium marriage exhibits block matching, which is weakly positive assortative.⁸ In Smith (2006), search is also completely random. His focus is on how the functional form of the output function of marriage affects the equilibrium matching pattern. JT extend BC by allowing agents to choose with whom to meet. That is, men and women are free to create submarkets beforehand. They show that this possibility makes the matching pattern more assortative (but not perfectly assortative). The feature that agents can choose who to meet with is related to the targeted search in our model. The difference is that in our targeted search the submarkets are exogenously fixed (defined by women types), and only men choose which types of women to meet with. Other differences between our model and results and those of the aforementioned three papers will be elaborated on later in the text.

Burdett and Wright (1998) studies a search/matching model of the marriage market with horizontally differentiated men and women, which is also modeled as an i.i.d. match value. However, in their model there is no vertical differentiation. In the existing literature on search/matching in the marriage market, only Sundaram (2000) and Herrenbrueck et al. (2016) combine both vertically and horizontally differentiated men and women.⁹ However, in Sun-

⁷A non-exhaustive list includes: Burdett and Coles (1997), Burdett and Wright (1998), Eeckhout (1999), Bloch and Ryder (2000), Shimer and Smith (2000), Smith (2006), and Jacquet and Tan (2007).

⁸Eeckhout (1999) extends BC to more general preferences. Bloch and Ryder (2000) and Burdett et al. (2004) consider the coexistence of a decentralized market and a centralized matching market organized by a matchmaker, who charges fees. Ong et al. (2017) empirically study China’s marriage market in which women prefer men who have higher incomes than themselves.

⁹Hitsch et al. (2010a) is an empirical paper that combines both vertical and horizontal differentiations, in

daram (2000) the horizontal heterogeneity is quite special and limited. In Herrenbrueck et al. (2016), the vertical types and the horizontal match value are normally and independently distributed. They show the existence and uniqueness of equilibrium; but they are not able to provide a clean characterization of the equilibrium or carry out comparative statics.¹⁰ Another important difference is that in both papers search is completely random, but in our model search is targeted.¹¹

The targeted search adopted in our paper is related to Moen’s (1997) concept of competitive search equilibrium in the labor market context.¹² In his model, workers are endogenously segmented into different submarkets. Each submarket has the same type of firms, but across submarkets, the firm types are different. More closely related to our paper is Yang (2015). He develops a model of targeted search in a labor market setting, and shows that the Internet might have contributed to rising wage inequality as well as wage polarization. Similar to the current model, in his model firms of different types constitute distinctive submarkets, and workers can choose beforehand in which submarket to participate, but within each submarket search is random.¹³ The current paper differs from Yang (2015) in two crucial aspects. First, in his model there is wage bargaining and thus it is a model with transferable utility, while our marriage model has non-transferable utility. Second, in his model there is no horizontal differentiation, whereas in our model agents differ in the vertical dimension as well as in the horizontal dimension.

The targeted search in our model in some aspects is similar to partially directed search (Menzio, 2007; Lester, 2011), as men can direct their search to a particular submarket but within submarkets search is random. We adopted the terminology of targeted search because no party posts any contracts beforehand in our model, which is a distinguishing feature of the models of directed search. The terminology of targeted search was also used in Yang (2013) as well as in Cheremukhin et al. (2016), but these search protocols are used in different contexts and are very different from ours.

The rest of the paper is organized as follows. Section 2 sets up the model. In Section 3 we characterize the market equilibrium and the endogenous segmentation of men into different submarkets. In Section 4 we study the impacts of an Internet-induced increase in search

which matching is frictionless.

¹⁰Herrenbrueck et al. (2016) resort to numerical simulation to estimate their model, using data in online dating markets.

¹¹Another related paper is Banerjee et al. (2013). They develop a marriage model of India, which incorporates vertical differentiation and agents’ preferences for Caste. In particular, the second component can be either horizontal (agents prefer marriages within the same Caste) or vertical (agents prefer “marrying up”). Their model is a matching model without search.

¹²Also see the directed search models, such as Mortensen and Wright (2002), Shimer (2005), Menzio (2007), and Eeckhout and Kircher (2010).

¹³In Bloch and Ryder (2000) the search technology is also not completely random.

efficiency, and Section 5 studies the comparative statics when the measures of women in some submarkets change. In Section 6 the impacts of an increase in the dispersion of the horizontal match value are analyzed, and Section 7 considers the possibility of horizontal targeting. Section 8 offers a conclusion and some discussion. All the proofs are provided in the appendix.

2 Model

Consider a marriage market with heterogeneous men and heterogeneous women. Each man is characterized by his type y , which is continuously distributed on $[y, \bar{y}]$ ($y > 0$) with cumulative distribution function $F(y)$ and density function $f(y)$. The measure of men is normalized to 1. There are N types of women, and type n is characterized by θ_n , $\theta_1 > \theta_2 \dots > \theta_N > 0$.¹⁴ The measure of type n women is X_n , which is exogenously fixed. The total measure of women is $\sum X_n = X$, which is assumed to be close to 1. Sometimes we call type n women as n th class women. Time is continuous and all agents have the common discount rate r .

If a man i of type y and a woman j of type n marry, each of them enjoys the same flow payoff $\varepsilon_{ij}\theta_n y$. The term $\theta_n y$ is the basic productivity of the marriage, which is supermodular in the man's and woman's type.¹⁵ The term ε_{ij} is the match fitness, which captures the horizontal aspects of matching (the compatibility of temperaments, personalities, etc.). Note that ε_{ij} and the basic productivity $\theta_n y$ are supermodular as well.¹⁶ The match fitness ε is i.i.d. across all man-woman pairs, and is independent of men's or women's types. In particular, ε is distributed on $[1 - \gamma, 1 + \gamma]$, $\gamma \in (0, 1)$, with cumulative distribution function $G(\varepsilon)$, density function $g(\varepsilon)$, and $E(\varepsilon) = 1$. Since $\gamma < 1$, the flow payoff of a marriage between any man and any woman is always positive. If an agent is single, his/her flow payoff is 0.

Men actively search for women and the search is not random.¹⁷ In particular, the marriage market is segmented into N submarkets, with each type of women constituting a distinctive submarket. The identity of each submarket (and hence women's types) is publicly observable. One can imagine that women of different classes go to different bars or attend different clubs. Each type of men can target their search by deciding in which submarket or submarkets to participate. If a man decides to participate in several submarkets, then he has to allocate his search efforts across these submarkets.

As will be shown later, generally each type of men will only search in one submarket,

¹⁴In section 3.3, we will discuss why we make women's types finite.

¹⁵The assumption that the basic productivity of a marriage is supermodular in y and θ_n is not essential for the qualitative results of the paper. What is required is that an agent's basic utility in a marriage is increasing in his/her partner's type, as in BC and JT.

¹⁶The analysis in this paper extends qualitatively to more general situations where the flow payoff is supermodular in the basic productivity and ε_{ij} .

¹⁷Women do not search for men. Thus it is a one-sided search model. In section 3.3 we will discuss this feature further.

which uniquely gives them the highest expected discounted utility. Let Y_n be the measure of single men and $q_n = Y_n/X_n$ be the expected queue length in submarket n . We assume that the matching function is generated by an urn ball technology: at any instant the number of meetings in submarket n is αY_n . Equivalently, men's contact rate for women in any submarket is α ,¹⁸ while a type n woman's contact rate with men is αq_n , which is increasing in q_n (Mortensen and Pissarides, 1999, p. 2575-2576). One can think of α as men's common search intensity, which is exogenously given. The urn ball matching technology means that there is no direct externality of search congestion among men,¹⁹ and we will further discuss this matching technology in the Conclusion.

The search within each submarket is random. That is, when a type n woman meets a man, the man's type is a random draw from $F_n(y)$ (the density function is denoted as $f_n(y)$), which is the distribution of men's types active in submarket n . Once a man and a woman meet, they observe each other's type and the match value ε immediately. Then they simultaneously decide whether to marry. A marriage is consummated if and only if both agree to marry. Once married, they are out of the market forever, and they are replaced by clones of the corresponding types.²⁰ Due to the fact that all men have the same contact rate α and the clone-replacement assumption, the density function $f_n(y)$ is inherited from the original density f , $f_n(y) = \frac{f(y)}{Y_n}$ for any y on the support.

We label the specified search protocol as targeted search. This is because the search is partially directed, since men can choose which submarket(s) or which type(s) of women to target beforehand, and partially random, as it is within each submarket. In the existing literature of matching in the marriage market, search is either random (BC, Smith 2006, etc.), or both men and women can endogenously form submarkets (JT). Given the presence of the horizontal differentiation, men are searching for right (compatible) women of the right (vertical) type, and the same applies to women.

3 Market Equilibrium

3.1 Preliminary Analysis

A type y man's strategy consists of two parts: a participation strategy as to in which submarket(s) to participate, and a matching strategy as to which set of women (in terms of both n

¹⁸If a man allocates his search efforts in several submarkets according to $\sigma = \{\sigma_n\}$, where $\sigma_n \geq 0$ and $\sum \sigma_n = 1$, then his contact rate in submarket n is $\alpha \sigma_n$.

¹⁹Shimer (2005) also uses urn ball matching technology, but there is direct search congestion. The difference arises because our model is in continuous time, in which all meetings are bilateral, while his model is static and thus meetings are n to 1.

²⁰This is a simplifying assumption, making the distribution of single men the same as the original distribution F . A further discussion will be offered in the Conclusion.

and the match value ε) to accept. A type n woman's (matching) strategy is a decision rule as to which set of men (in terms of both y and ε) to accept.

A participation strategy profile of all men leads to a segmentation of men into N submarkets. Denote a segmentation as $\bar{S} : [y, \bar{y}] \rightarrow \{1, \dots, N\}$. Let y_n be the set of men types participating in submarket n . Thus, $\{y_n\}_{n=1}^N$, which exhausts men's type space $[y, \bar{y}]$, also represents a segmentation. Notice here the segmentation allows a man to target multiple submarkets.

Given the clone-replacement assumption, once a segmentation $\{y_n\}_{n=1}^N$ is determined, the measure of single men active in submarket n , Y_n , and its distribution $F_n(y)$ are both determined. The market condition in submarket n is thus summarized by $\{q_n, F_n(y)\}$. Denote $U_n(y)$ as a man's expected discounted utility who is of type y and participates in submarket n , and denote V_n as a type n woman's expected discounted utility. Since the search environment is stationary, the optimal matching strategy of a type y man participating in submarket n is characterized by a reservation utility: accept a woman if and only if the overall matching utility is greater than his continuation value $U_n(y)$, or equivalently, if and only if the match value $\varepsilon \geq \min\{\max\{\frac{U_n(y)}{\theta_n y}, 1 - \gamma\}, 1 + \gamma\} \equiv \hat{\varepsilon}_n^m(y)$. Similarly, a type n woman's optimal strategy is also characterized by a reservation match value $\hat{\varepsilon}_n^w(y)$: a man of type y is accepted if and only if the match value $\varepsilon \geq \min\{\max\{\frac{V_n}{\theta_n y}, 1 - \gamma\}, 1 + \gamma\} \equiv \hat{\varepsilon}_n^w(y)$. Note that $\hat{\varepsilon}_n^w(y)$ is weakly decreasing in y , as a higher type man is more productive in marriage. Since a consummated marriage needs both parties to say yes, the acceptance cutoff (in terms of the horizontal match fitness) that ensures matching between a type y man and a type n woman is $\hat{\varepsilon}_n(y) = \max\{\hat{\varepsilon}_n^m(y), \hat{\varepsilon}_n^w(y)\}$.

The value function $U_n(y)$ can be written as

$$rU_n(y) = \alpha \int_{\hat{\varepsilon}_n^w(y)}^{1+\gamma} \max\{\varepsilon \theta_n y - U_n(y), 0\} dG(\varepsilon).$$

The constraint $\varepsilon \geq \hat{\varepsilon}_n^w(y)$ reflects the fact that a marriage is consummated only if the woman involved agrees. Equivalently, in terms of $\hat{\varepsilon}_n^m(y)$, it can be written as

$$U_n(y) = \max_{\hat{\varepsilon}_n^m(y)} \frac{\alpha \theta_n y \int_{\hat{\varepsilon}_n^m(y)}^{1+\gamma} \varepsilon dG(\varepsilon)}{r + \alpha(1 - G(\hat{\varepsilon}_n^m(y)))} \text{ subject to } \hat{\varepsilon}_n^m(y) \geq \hat{\varepsilon}_n^w(y). \quad (1)$$

Assumption 1: $\gamma > \frac{r}{r+\alpha}$.

Lemma 1 *All men's optimal reservation match values are the same, regardless of their types and their chosen submarkets: $\hat{\varepsilon}_n^m(y) = \hat{\varepsilon}^m \in [1 - \gamma, 1 + \gamma]$. Moreover, if Assumption 1 is satisfied, then $\hat{\varepsilon}^m > 1 - \gamma$ and it strictly increases in α .*

Assumption 1 ensures that men care enough about the horizontal match fitness so that they will not settle for women with the lowest match fitness $\varepsilon = 1 - \gamma$, which is true if the

lowest match fitness is bad enough (γ is large) or the cost of rejection is small enough (α is large or r is small). From now on, we will always impose Assumption 1 and hence focus on interior $\widehat{\varepsilon}^m$ only. All of the analysis and results can be extended to the case with $\widehat{\varepsilon}^m = 1 - \gamma$.

The result that $\widehat{\varepsilon}^m$ is constant across all submarkets for all men stems from the fact that in the flow payoff of marriage the horizontal and vertical components are multiplicative. That is, men's value function can be written as $z_1(\theta_n, y)z_2(\varepsilon)$ with some functions z_1 and z_2 . Given this functional form, in each submarket n maximizing the value is equivalent to maximizing $z_2(\varepsilon)$ by choosing ε , and thus all men have the same optimal $\widehat{\varepsilon}^m$.²¹ The lemma also shows that a higher contact rate α reduces the cost of rejection, thus men become more choosy, leading to a higher $\widehat{\varepsilon}^m$.

Denote $\overline{U}_n(y)$ as a type y man's maximum value in submarket n , which is achieved when type n women accept him whenever $\varepsilon \geq \widehat{\varepsilon}^m$. By (1), we have

$$\overline{U}_n(y) = \frac{\alpha \theta_n y \int_{\widehat{\varepsilon}^m}^{1+\gamma} \varepsilon dG(\varepsilon)}{r + \alpha(1 - G(\widehat{\varepsilon}^m))} = \frac{\alpha \theta_n y (1 - G(\widehat{\varepsilon}^m)) E[\varepsilon | \varepsilon \geq \widehat{\varepsilon}^m]}{r + \alpha(1 - G(\widehat{\varepsilon}^m))}. \quad (2)$$

Given type n women's value function V_n , we can trace the actual acceptance set of men for type n women: $\widehat{\varepsilon}_n(y) = \max\{\widehat{\varepsilon}_n^w(y), \widehat{\varepsilon}^m\}$. Note that $\widehat{\varepsilon}_n(y)$ is weakly decreasing in y . For higher types in submarket n , it might reach the lower bound $\widehat{\varepsilon}^m$. Given the set of male types active in submarket n , y_n , we can write down V_n as

$$rV_n = \alpha q_n \int_{y_n} \int_{\widehat{\varepsilon}_n(y)}^{1+\gamma} \{\varepsilon \theta_n y - V_n\} dG(\varepsilon) dF_n(y).$$

Rearranging, we get

$$V_n = \frac{\alpha \int_{y_n} \int_{\widehat{\varepsilon}_n(y)}^{1+\gamma} \theta_n y \varepsilon dG(\varepsilon) dF_n(y)}{r/q_n + \alpha \int_{y_n} [1 - G(\widehat{\varepsilon}_n(y))] dF_n(y)}. \quad (3)$$

Following the previous analysis, now we provide a definition of the equilibrium.

Definition 1 A (marriage) market equilibrium consists of a segmentation \overline{S} or $\{y_n\}_{n=1}^N$, men's matching strategy $\widehat{\varepsilon}^m$, and women's matching strategy $\{\widehat{\varepsilon}_n^w(y)\}_{n=1}^N$ such that:

(i) Given women's optimal matching strategy $\{\widehat{\varepsilon}_n^w(y)\}_{n=1}^N$, for each type of man y the optimal matching strategy is given by $\widehat{\varepsilon}^m$.

(ii) Given \overline{S} or $\{y_n\}_{n=1}^N$, and men's optimal matching strategy $\widehat{\varepsilon}^m$, women's matching strategy $\{\widehat{\varepsilon}_n^w(y)\}_{n=1}^N$ is optimal for all n .

²¹If the flow payoff is additive in the horizontal and vertical components, say $\theta_n y + \varepsilon$, then different types of men in the same submarket (as well as the same type of men participating in different submarkets) will choose different $\widehat{\varepsilon}^m$. But this will greatly complicate the analysis.

(iii) Given \bar{S} or $\{y_n\}_{n=1}^N$, and women's optimal matching strategy $\{\hat{\varepsilon}_n^w(y)\}_{n=1}^N$, any type y man has no incentive to deviate to another submarket: for any y , $y \in y_n$, $U_n(y) \geq U_{n'}(y)$ for any $n' \neq n$.

Lemma 2 In any market equilibrium, (i) $\frac{V_n}{\theta_n}$ is weakly decreasing in n ; (ii) $\hat{\varepsilon}_n(y)$ is weakly decreasing in n for any y .

Lemma 2 is intuitive. For a type y man to have an incentive to participate in a lower submarket n , his probability of being accepted by all higher class women must be lower than his probability of being accepted by class n women. This means that in equilibrium higher class women must be pickier, or $\hat{\varepsilon}_n(y)$ is decreasing in n , which further implies that $\frac{V_n}{\theta_n}$ should be decreasing in n . This pattern also suggests that, in equilibrium, higher type men participate in weakly higher submarkets, or weakly positive assortative matching. To formally establish this single crossing property, we need an additional assumption regarding the distribution of ε .

Assumption 2: The density $g(\varepsilon)$ is logconcave and satisfies

$$\frac{g'(1+\gamma)}{g(1+\gamma)} + \frac{1}{1+\gamma} \geq 0. \quad (4)$$

This assumption will be maintained throughout the paper. The assumption that $g(\varepsilon)$ is logconcave is standard in the literature, and it is satisfied by many distributions. Since $g(\varepsilon)$ is logconcave, $g'(\varepsilon)/g(\varepsilon)$ is monotonically decreasing in ε .²² Thus $g'(\varepsilon)/g(\varepsilon)$ reaches minimum when $\varepsilon = 1 + \gamma$. Condition (4) ensures that $g'(\varepsilon)/g(\varepsilon)$ is not too negative for all ε , i.e., the density does not decrease too rapidly (relatively). This assumption guarantees that the reduction in the probability of being accepted from switching to a higher submarket is smaller for a higher type man, as will become clear later.

Note that condition (4) is a weak condition. For uniform distribution, it is trivially satisfied. Now consider truncated normal distribution on support $[1-\gamma, 1+\gamma]$, with mean 1 and variance σ^2 :

$$g(\varepsilon) = \frac{\phi\left(\frac{\varepsilon-1}{\sigma}\right)}{\Phi\left(\frac{\gamma}{\sigma}\right) - \Phi\left(\frac{-\gamma}{\sigma}\right)} \frac{1}{\sigma},$$

where ϕ and Φ are the p.d.f. and c.d.f. of a standard normal distribution, respectively. In this case, $\frac{g'(1+\gamma)}{g(1+\gamma)} = -\frac{\gamma}{\sigma^2}$. Thus condition (4) boils down to $\frac{\sigma^2}{\gamma(1+\gamma)} \geq 1$, or the variance σ^2 is big enough.

²²See Bagnoli and Bergstrom (2005).

Lemma 3 Consider two submarkets n' and n , with $n' < n$, and two types of men y and y' with $y' > y$. The following scenario cannot occur in any market equilibrium: type y men participating in submarket n' , type y' men participating in submarket n , and $U_n(y) > 0$.

Lemma 3 implies that equilibrium segmentation must exhibit weakly positive assortative matching.²³ Therefore, we have proved the following proposition.

Proposition 1 In any market equilibrium, a higher type man must participate in a weakly higher submarket.

The underlying reasons for the weakly positive assortative matching (block matching) is that the reduction in the probability of being accepted by choosing a higher class woman is smaller for a higher type man. To see this, suppose $\hat{\varepsilon}_n(y) = \frac{V_n}{\theta_n y} > \hat{\varepsilon}^m$. Then it is immediate that

$$\hat{\varepsilon}_{n-1}(y) - \hat{\varepsilon}_n(y) = \frac{1}{y} \left(\frac{V_{n-1}}{\theta_{n-1}} - \frac{V_n}{\theta_n} \right)$$

is decreasing in y . Given that the distribution of ε is not too irregular (Assumption 2 ensures this), the property that $\hat{\varepsilon}_{n-1}(y) - \hat{\varepsilon}_n(y)$ is decreasing in y implies that switching from class n women to class $n - 1$ women leads to a smaller reduction in the probability of being accepted for a higher type man. This means that, relative to a lower type man, it is less costly for a higher type man to choose a higher class woman.

Given the result of Proposition 1, an equilibrium segmentation is characterized by a non-increasing sequence $\{\hat{y}_n\}$, such that all men with $y \in [\hat{y}_n, \hat{y}_{n-1}]$ participate in submarket n . Moreover, the marginal type \hat{y}_n is indifferent between submarket n and submarket $n + 1$.²⁴ This is possible because, being the lowest type in submarket n , he has a lower chance of being accepted, while in submarket $n + 1$ he is the highest type, and thus has a higher chance of being accepted. Now we can explicitly write $F_n(y)$, the type distribution of men in submarket n . Specifically, $F_n(y) = \frac{F(y) - F(\hat{y}_n)}{F(\hat{y}_{n-1}) - F(\hat{y}_n)}$ for $y \in [\hat{y}_n, \hat{y}_{n-1}]$, and $Y_n = F(\hat{y}_{n-1}) - F(\hat{y}_n)$.

Remark 1 Although this is a model of non-transferrable utilities, the presence of horizontal differentiation leads to different accepting probabilities for different types of men within the same submarket, which in some sense act as shadow prices.

²³The lemma only applies to men whose discounted payoff is strictly positive. It is possible that there exists some y (very low types) such that $U_N(y) = 0$ (never accepted by any woman). Then $U_n(y) = 0$ for any n , and these types of men are indifferent among all submarkets in equilibrium. Does that mean that an equilibrium exists that is not weakly positive assortative? Not really. This is because these types of men are always rejected in any submarket. If we only focus on the set of men that have strictly positive probabilities of being accepted by some women, then the equilibrium is still weakly positive assortative.

²⁴The proposition shows that, for a generic type of man (except for the cutoff types y_n^*), there is a unique submarket which gives him the highest expected payoff, and thus he will only participate in that submarket.

Based on the analysis so far, a market equilibrium is characterized by a nonincreasing sequence of cutoff types $\{y_n^*\}$ such that:

(i) Given $\{y_n^*\}$, for any $n = 1, \dots, N$: V_n satisfies equation (3), $\hat{\varepsilon}_n^w(y) = \frac{V_n}{\theta_n y}$, and $\hat{\varepsilon}_n(y) = \max\{\hat{\varepsilon}^m, \hat{\varepsilon}_n^w(y)\}$ (women's matching strategies are optimal).

(ii) Given $\hat{\varepsilon}_n^w(y)$, the following indifference conditions are satisfied for the marginal types: for all $n = 1, \dots, N - 1$

$$\begin{aligned} U_n(y_n^*) &= U_{n+1}(y_n^*) \text{ if } y_n^* > \underline{y}; \text{ (interior solution)} \\ U_n(y_n^*) &\geq U_{n+1}(y_n^*) \text{ if } y_n^* = \underline{y}. \text{ (corner solution)} \end{aligned} \quad (5)$$

More explicitly, the indifference condition (5) can be written as

$$\frac{\theta_n [1 - G(\hat{\varepsilon}_n(y_n^*))] E[\varepsilon | \varepsilon \geq (\hat{\varepsilon}_n(y_n^*))]}{r + \alpha [1 - G(\hat{\varepsilon}_n(y_n^*))]} = \frac{\theta_{n+1} [1 - G(\hat{\varepsilon}_{n+1}(y_n^*))] E[\varepsilon | \varepsilon \geq (\hat{\varepsilon}_{n+1}(y_n^*))]}{r + \alpha [1 - G(\hat{\varepsilon}_{n+1}(y_n^*))]}. \quad (6)$$

Since $\theta_n > \theta_{n+1}$, the indifference condition (5) means that $\hat{\varepsilon}_{n+1}(y_n^*) < \hat{\varepsilon}_n(y_n^*)$. That is, the marginal type y_n^* must have a strictly higher accepting probability in submarket $n + 1$ than in submarket n .²⁵

The following lemma is useful in later analysis.

Lemma 4 (i) Fixing the upper bound \hat{y}_{n-1} , as the lower bound \hat{y}_n decreases, V_n will weakly increase: V_n will strictly increase if $\hat{\varepsilon}_n(\hat{y}_n) < 1 + \gamma$, and V_n will remain the same if $\hat{\varepsilon}_n(\hat{y}_n) = 1 + \gamma$. (ii) Fixing the lower bound \hat{y}_n , as the upper bound \hat{y}_{n-1} increases, V_n will strictly increase.

Lemma 4 actually points out the driving force behind equilibrium segmentation. That is, men within the same submarket impose indirect negative externalities on each other, through the channel of changing women's payoff and their matching strategy. With more men in a particular submarket, it increases the meeting rate and the expected payoff of the women in that submarket. As a result, they become more picky about men, which means that men's expected payoff of participating in this submarket at least weakly decreases.²⁶ Another observation is that adding higher types of men to a given submarket has a bigger impact than adding lower types. This is because higher type men are more desirable to women, and thus will boost women's expected payoff more significantly.

²⁵The proof of Lemma 1 also shows that the term $\frac{[1 - G(\hat{\varepsilon})] E[\varepsilon | \varepsilon \geq (\hat{\varepsilon})]}{r + \alpha [1 - G(\hat{\varepsilon})]}$ is strictly decreasing in $\hat{\varepsilon}$ for $\hat{\varepsilon} \geq \hat{\varepsilon}^m$.

²⁶Type y men's expected payoff strictly decreases if $\hat{\varepsilon}_n(y)$ strictly increases or women's acceptance cutoff binds after the change.

3.2 Existence and Uniqueness of Equilibrium

The indifference condition (6) is a second order difference equation, which is highly nonlinear. Therefore, we have to establish the existence and uniqueness of equilibrium by ourselves. We proceed by induction.

Fix the lower bound \hat{y}_n , and consider partial equilibrium in submarkets $i = 1, \dots, n$: a segmentation of men with types $y \in [\hat{y}_n, \bar{y}]$ into submarkets $i = 1, \dots, n$ such that the equilibrium conditions (5) are satisfied. We denote the partial equilibrium segmentation as $\{y_i^*(\hat{y}_n)\}_{i=1}^{n-1}$. And sometimes we abuse notation and simply write it as $\{y_i^*\}_{i=1}^{n-1}$.

Lemma 5 *Fix any $\hat{y}_2 \in [\underline{y}, \bar{y}]$. (i) There is a unique $y_1^* \in [\hat{y}_2, \bar{y}]$ achieving partial equilibrium in submarkets 1 and 2. (ii) y_1^* is weakly increasing in \hat{y}_2 . (iii) V_2^* is weakly decreasing in \hat{y}_2 , and it is strictly decreasing in \hat{y}_2 if y_1^* is strictly increasing in \hat{y}_2 .*

Lemma 6 *Suppose the properties in Lemma 5 hold for n . That is, given any $\hat{y}_n \in [\underline{y}, \bar{y}]$, we have: (i) there is a unique $\{y_i^*\}_{i=1}^{n-1} \in [\hat{y}_n, \bar{y}]$ achieving partial equilibrium in submarkets $1, \dots, n-1$; (ii) y_{n-1}^* is weakly increasing in \hat{y}_n ; (iii) V_n^* is weakly decreasing in \hat{y}_n , and it is strictly decreasing in \hat{y}_n if y_{n-1}^* is strictly increasing in \hat{y}_n . Then, given any $\hat{y}_{n+1} \in [\underline{y}, \bar{y}]$, these properties also hold for $n+1$.*

Lemma 5 shows that Lemma 6 holds for $n = 2$. Applying Lemma 6 repeatedly, the results hold for any $n \leq N$. Thus, we have proved the following proposition.

Proposition 2 *There is a unique market equilibrium.*

By Lemma 5 and Lemma 6, we also have the following useful corollary.

Corollary 1 *(i) If \hat{y}_n , $2 \leq n \leq N-1$, increases for exogenous reasons, then $\{y_i^*\}_{i=1}^{n-1}$, which ensures partial equilibrium in submarkets $1, \dots, n$, all weakly increase, and V_n weakly increases as well. (ii) If \hat{y}_n , $1 \leq n \leq N-2$, increases for exogenous reasons, then $\{y_i^*\}_{i=n+1}^{N-1}$, which ensures partial equilibrium in submarkets $n+1, \dots, N$, all strictly increase.*

While part (i) of Corollary 1 is directly implied by Lemma 5 and Lemma 6, part (ii) can be proved in a similar way by reversing the direction. Roughly speaking, this corollary means that all equilibrium cutoffs must move in the same direction.

The existence of equilibrium is not surprising, as each man has to choose some submarkets to participate in. The uniqueness of the equilibrium is due to the indirect negative externalities that men within the same submarket impose on each other. Having more men in a particular submarket increases women's expected payoff, and they become more choosy. Thus, men's

probabilities of being accepted weakly decrease (strictly, for men of low types), which reduces their expected payoff in this particular submarket. In short, more men in a submarket reduces the attractiveness of this submarket to men. On the other hand, more men in a submarket means fewer men in other submarkets. For the same reason, other submarkets become more attractive to men. This indirect negative externality means that in equilibrium the segmentation has to be right to ensure no man has an incentive to deviate to another submarket, which implies the uniqueness of equilibrium.²⁷

3.3 Equilibrium features

The market equilibrium exhibits endogenous segmentation in the vertical dimension: they are endogenously segmented into N (may be fewer) submarkets, with men in submarket n only marrying the corresponding n th class women. Note that the set of men active in the lower submarkets might be empty, because, all other things being equal, higher type women are more desirable to men.²⁸ This means that women in the lower submarkets might never get married, a feature similarly noted in BC.

The market equilibrium also exhibits the following features that are not present in the existing models, mainly because our model incorporates horizontal differentiation. The first feature is about the marrying speeds. Within each submarket, men with higher vertical types are (weakly) faster to marry than the lower types. This is because, though each type of man has the same contact rate, a higher type man has a higher chance to be accepted by women ($\widehat{\varepsilon}_n^w(y)$ is decreasing in y). However, across submarkets men's marrying speed is non-monotonic in vertical types. In particular, there is an upward jump in the marrying speed when we move from a higher submarket to a lower submarket. Thus, a higher type man in a higher submarket can have a lower marrying speed than a lower type man in a lower submarket.

The second feature is about sorting in the horizontal dimension. Within each submarket, since women's acceptance cutoff for a lower type man is higher, among the married couples the compatibility of men and women (in expectation) is higher in the horizontal dimension if a man's vertical type is lower. Again, across submarkets this monotonicity no longer holds: moving from a higher submarket to a lower submarket, the compatibility of couples in the horizontal dimension has a downward jump. The third feature is that, within each submarket,

²⁷More formally, consider submarkets n and $n+1$ and the marginal type y_n^* . If the marginal type y_n^* increases, then, due to the indirect negative externality, V_n decreases and V_{n+1} increases. Thus $U_n(y_n^*)$ increases but $U_{n+1}(y_n^*)$ weakly decreases, meaning that now a type y_n^* man strictly prefers submarket n , which contradicts the fact that the marginal type increases. For a similar reason, there is no equilibrium with a marginal type lower than y_n^* .

²⁸If submarket n has a positive measure of men, then each higher submarket i , $i < n$, must be active, or have a positive measure of men. Similarly, if submarket n is inactive or has no men participating in it, then all lower submarkets ($i > n$) must be inactive as well.

the higher types of men get higher expected payoffs than the lower types. This is, again, because a higher type man has a higher chance of being accepted.

The first and third features are different from those in BC and JT, while the second feature is absent in their models. Specifically, in BC and JT men in the same submarket have the same expected payoff and the same matching rate. The differences arise mainly because in their models there is no horizontal match value. As a result, each type of men in the same submarket is always accepted by women of the corresponding submarket, which leads to the same payoff for men in the same submarket.²⁹

Finally, although the expected payoffs of men and women in a submarket only depend on the market conditions within the same submarket (q_n and $F_n(y)$), all the submarkets are interlinked. This is because which set of men participate in a particular submarket depends on the market conditions in the adjacent submarkets. Specifically, between any two adjacent submarkets the marginal type of man has to be indifferent between the two submarkets. As a result, all submarkets are indirectly linked. A related feature is that men's equilibrium payoff schedule (as a function of type), $U_e(y)$, is continuous over types, as the marginal types are indifferent between adjacent submarkets.

These features mentioned in the last paragraph are again different from those discovered by BC and JT. In particular, in their models, men's equilibrium payoff schedule is discontinuous across submarkets, with the marginal types strictly preferring the higher submarket to the adjacent lower submarket. The underlying reason is again that in their models there is no horizontal match value. Actually, in both models the submarkets (classes) are determined from top to bottom: a lower class man will not be accepted by a higher class woman, even if he strictly prefers the higher class woman. This also means that in their models there is only one-directional linkage between classes: only higher classes affect lower classes. If the market conditions in the lower classes change, they will not affect the segmentation among the higher classes. In contrast, in our model changes in the market conditions in lower classes (submarkets) could affect the segmentation among higher classes (if $U_{n+1}(y_n^*) < \bar{U}_{n+1}(y_n^*)$).

We want to point out that the distinguishing feature of our model is that it incorporates horizontal differentiation. To see this, suppose there is no horizontal differentiation, say ε is degenerate at 1, but search is still targeted. The market equilibrium in this setting is qualitatively very similar to those of BC and JT. In particular, n th class women are indifferent between accepting and rejecting the bottom marginal type of n th class men, and that marginal type of men strictly prefers submarket n to the adjacent lower submarket $n + 1$.

Is it the case that in equilibrium the expected queue length q_n^* must be higher in a higher submarket? The general pattern is yes. Recall that to make a marginal type of men indifferent

²⁹Another difference is that in BC and JT, an agent's flow payoff in a marriage does NOT depend on his/her own type, but only depends on his/her partner's type.

between two adjacent submarkets, his probability of being accepted by the higher class women must be strictly lower than his probability of being accepted by the lower class women. A higher q_n^* in the higher submarket, by increasing V_n^*/V_{n+1}^* , would help in achieving the indifference condition.

Discussion Here we explain why we made some special modeling assumptions. The first special assumption we made is that women do not actively search. The main reason for making this assumption is technical tractability. In BC or JT, since the acceptance decision is either 0 or 1, the classes can be determined from top to bottom: by solving the reservation type of the top agents, the bottom type of the top class is determined. Applying the same procedure recursively, the endogenous segmentation of all classes can be determined. In our model, the presence of horizontal differentiation implies that the segmentation of lower classes affects the segmentation of higher classes as well. As a result, solving for the equilibrium is much more complicated. If women can also actively search and target men’s vertical type, then the model becomes intractable.³⁰

Another justification for this assumption is that it is realistic in traditional societies. In modern societies, women actively search to some extent. But, evidence in sociology studies shows that men are still more likely to take initiatives in dating. Peplau et al. (1993) discover that men holding the more traditional sex role ideology take more initiatives in the couple’s first getting to know each other whereas there is no significant relationship for women. They also find that majority of men in their year 1972 sample are traditional (46% of them can be classified as traditionalists and 31% of them moderates.), and hence indeed men on average take more initiatives.³¹ Recent empirical evidence on online dating (Hitsch et al., 2010a, 2010b) also shows that men are more active in searching than women. For instance, Hitsch et al. (2010b) report that the number of profiles viewed by a man on average is about three times of that of a woman (3702 men browsed 597,169 profiles while 2783 women only browsed 196,363 profiles). Moreover, men are roughly 40% more likely to initiate contact with a woman after viewing her profile than vice versa (12.5% vs. 9.0%). Overall, an average man sent 3.2 times more first contact e-mails than did an average women. Compared to men, women are more likely to favor the passive path over the assertive path.

The second special assumption is that while men’s types are continuous, women’s types are finite. This is another technical assumption which makes the model tractable. Consider the

³⁰If women can also actively search and target men’s vertical types, then low types of women can ensure a positive marriage probability by targeting low types of men. Moreover, matching in the vertical dimension will become more assortative.

³¹The conventional male dominance ideology is still prevalent nowadays according to Pepin and Cotter (2018). They show that since mid-1990s, in fact, more and more men and women think that husband should make all family decisions.

case that women's types are also continuous, and their types are exogenously partitioned into N intervals, with each interval consisting of a separate submarket. In this case, in each submarket there are heterogeneous types of women. With the presence of horizontal differentiation, within a given submarket one needs to trace the acceptance set (in two dimensions) of each type of man and that of each type of woman. Moreover, men's acceptance sets and women's acceptance sets interact with each other, which means that it is very hard to pin down the equilibrium acceptance strategy for each type of men and for each type of women. This makes the model intractable.

Instead, by assuming that women's types are finite, women become homogeneous in each submarket. In this setting, targeted search separates a matching process with two-sided (vertical) heterogeneity into a two-stage process. Specifically, in the second stage, within each submarket it becomes a matching problem with one-sided heterogeneity, as women's side is homogeneous. In the first stage, it is again a matching problem with one-sided heterogeneity, as only men choose in which submarket to participate. This separation makes the model tractable. For the same reason, in this setting actually targeted search is more tractable than random search. This is because with random search one has to solve a matching process with two-sided (vertical) heterogeneity simultaneously.

With that said, the current model can approximate the case that women's types are continuous by increasing the number of women's types and letting it go to infinity. In this limiting case, in the vertical dimension weakly assortative matching converges to perfectly assortative matching. That is, for each type of women there is only one type of men targeting it.³² In the horizontal dimension, since men targeting each type of women is (almost) homogeneous in the limiting case, the cutoff of ε above which a match is consummated is unique in each submarket and it becomes (almost) continuous in women's type.

On the other hand, men's types being continuous is not essential. We can work out a model in which men's types are also finite. However, men's types being continuous simplifies our analysis, as otherwise we need to worry about the mixed strategies of the marginal types of men, who are indifferent between two adjacent submarkets.³³

Example 1 (*Benchmark*) Suppose $r = 0.05$, $\alpha = 0.25$. Men's types follow a truncated normal distribution on $[1, 5]$ with $\mu = 1.8$ and $var = 2$.³⁴ Women's type distribution is given by $[\theta_1, \theta_2, \theta_3, \theta_4, \theta_5] = [2, 1.6, 1.3, 1.1, 1]$, and $[X_1, X_2, X_3, X_4, X_5] = [0.05, 0.1, 0.15, 0.32, 0.3]$ (the

³²As women's types become finer, weakly assortative matching implies that the endogenous segmentation of men becomes finer as well.

³³In equilibrium, these types of men have to mix in the right way.

³⁴Both μ and var are the mean and variance of the original normal distribution before truncation. Men's type distribution here roughly resembles the income distribution in the U.S.

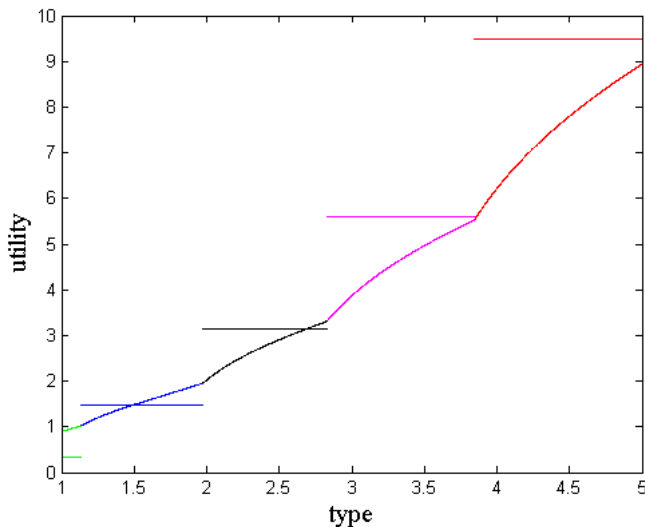


Figure 1: Men's Payoff Schedule and Women's Values

total measure of women $X = 0.92$).³⁵ The match value ε is uniformly distributed on $[1-\gamma, 1+\gamma]$ with $\gamma = 0.5$. The equilibrium is illustrated in Figure 1.

In Figure 1, the continuous curve indicates men's equilibrium payoff schedule (as a function of men's type), with different colors indicating men in different submarkets. The horizontal lines represent women's equilibrium values in different submarkets (the length corresponds to the type space of men participating in that submarket).³⁶ The key endogenous variables in the equilibrium are listed in Table 1.

Table 1: Equilibrium Variables

	Submarket 1	SM 2	SM 3	SM 4	SM 5
cutoff types (y_n^*)	3.8461	2.8281	1.9723	1.1374	—
q_n^*	3.2748	2.5052	1.7988	0.8539	0.1423

One prominent feature of Table 1 is that the equilibrium expected queue length q_n^* is higher in a higher submarket. In the highest three submarkets, the q_n^* 's are all significantly bigger than 1; in submarket 1, q_1^* is higher than 3, meaning that the men/women ratio is bigger than

³⁵The median type woman is a type 4 woman. The mean type lies between θ_4 and θ_3 . The distribution of women types is chosen to reflect the fact that it is skewed toward the low types.

³⁶Note that within a submarket a high type man's equilibrium value could be higher than women's equilibrium value. This is because women might match with men of lower types and have a lower marrying speed than a high type man.

3.³⁷ Because q_1^* is very big, V_1^* is higher than the equilibrium payoff of the highest type men (see Figure 1). This implies that in submarket 1 women's acceptance cutoff is binding for all types of men. The same pattern also holds in submarket 2: $V_2^* > U_2(y_1^*)$. On the other hand, there are very few men participating in submarket 5, though it has many women in it: q_5^* is less than 0.15. This leads to the feature that V_5^* is lower than the lowest type men's equilibrium utility.

The above observations imply that in equilibrium men are congested in higher submarkets.³⁸ In other words, the difference in q 's is the primary channel that makes the marginal types of men indifferent between two adjacent submarkets. Looking into the difference in the q 's, we find the following pattern: the difference is largest between q_3^* and q_4^* and decreases as we move to higher or lower submarkets. The underlying reason for this pattern is as follows. The men's type (recall $\mu = 1.8$) with the highest density is in submarket 4. Thus, the density function of men's types is decreasing in submarket 1, 2, and 3, hump-shaped in submarket 4, and increasing in submarket 5. Therefore, the averages of men's types in submarket 3 and 4 are relatively close. In order to restore the indifference condition by enlarging the difference between V_3 and V_4 , the difference between q_3^* and q_4^* has to be large enough. As we move to higher (or lower) submarkets, the bell-shaped density function implies that the difference in the averages of men's types in the adjacent submarkets is larger. This means that the indifference condition can hold even when the difference in q 's is smaller. In addition, because q_n^* is higher in a higher submarket, men's differences in equilibrium payoffs are relatively suppressed. For instance, $U_1(\bar{y})/U_5(\underline{y}) < 10 = \frac{\theta_1 \bar{y}}{\theta_5 \underline{y}}$. For the same reason, women's differences in equilibrium values are amplified: $V_1^*/V_5^* \simeq 25 > 10 = \frac{\theta_1 \bar{y}}{\theta_5 \underline{y}}$.

Men's and women's equilibrium marrying rates are illustrated in Figure 2, with the thick horizontal lines indicating women's marrying rates.³⁹ From the figure, we can see that women's marrying rates are higher in higher submarkets. This feature is mainly driven by the fact that q_n^* is higher in a higher submarket.⁴⁰ Moreover, this is consistent with the estimation result in Herrenbrueck et al. (2016, Figure 3); they found that the matching rate is monotonically increasing in agent's vertical component of attractiveness. For high types of men in submarkets 1 and 2, women's acceptance cutoff is always binding (since V_1^* and V_2^* are relatively big due to big q_1^* and q_2^*), and thus within each of these two submarkets men's marrying rate is

³⁷This means that type 1 women's contact rate is more than three times of the highest type men's contact rate.

³⁸This feature has something to do with the clone-replacement assumption. A more detailed discussion is offered in the Conclusion.

³⁹Specifically, a type y man's marrying rate in submarket n is $\alpha(1 - G(\hat{\varepsilon}_n(y)))$, and a type n woman's marrying rate is $\alpha q_n \int_{y_n} [1 - G(\hat{\varepsilon}_n(y))] dF_n(y)$.

⁴⁰Hitsch et al. (2010a, Figure 1) documented that physically more attractive women received more contact emails from men than less attractive women.

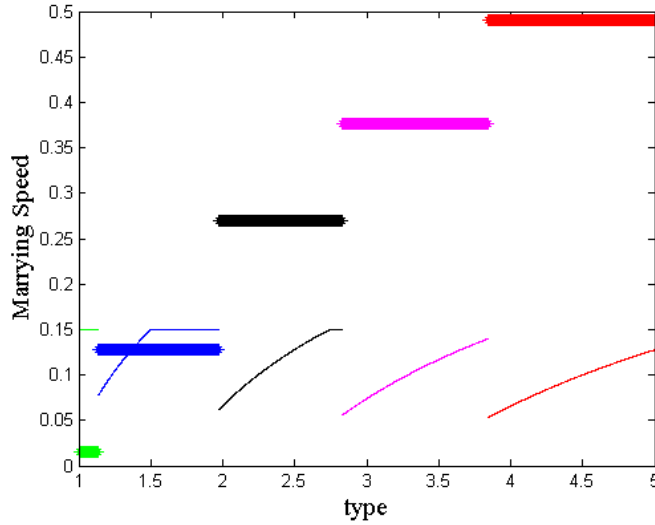


Figure 2: Men's and Women's Marrying Speed

monotonically increasing in men's types. In submarkets 3 and 4, men's acceptance cutoff is binding for high types of men. In submarket 5, men's acceptance cutoff is always binding, since V_5^* is relatively low due to a small q_5^* . Across submarkets, men's equilibrium marrying rate is not monotonic in type.

In the next three sections, we will conduct comparative statics, investigating how the equilibrium responds to shocks. To ease exposition, in the rest of the paper we assume the parameter values are such that in equilibrium each submarket of women is active or has a positive measure of men participating in it. Given that V_n is weakly decreasing in n , it is sufficient to assume that $V_N^* > 0$. Roughly speaking, it requires that the overall men/women ratio q is not too small.

4 An Internet-Induced Increase in Search Efficiency

The widespread use of the Internet reduces men's search costs for women. Instead of attending social gatherings, the Internet allows people to search and contact relevant partners at home. Given the exogenous fixed search intensity assumed in our model, the Internet increases each man's contact rate α . Note that the contact rates of all types of men increase by the same magnitude. We will conduct the analysis in two steps. In the first step, we hypothetically assume that the initial equilibrium segmentation does not change, and investigate how V_n and $U_n(y)$ will change. In the second step, we study how the endogenous vertical and horizontal sorting will adjust.

4.1 Fixed segmentation

Suppose α increases to $\alpha' > \alpha$, and the initial equilibrium segmentation \bar{S} does not change. We use superscript “ $'$ ” to denote the endogenous variables under α' . The first observation is that an increase in α will cause $\hat{\varepsilon}^m$ to increase to $\hat{\varepsilon}^{m'} > \hat{\varepsilon}^m$ (as shown in Lemma 1). Define the change in the equilibrium value of type y men in submarket n as $\Delta U_n(y) \equiv U_n'(y) - U_n(y)$.

To ease exposition, unless otherwise specified, in the analytical analysis of the rest of the paper, we further impose the assumption that for any n , there exists a neighborhood of y_{n-1}^* such that $\hat{\varepsilon}_n(y) = \hat{\varepsilon}^m$ for all y in the neighborhood, or $V_n < U_n(y_{n-1}^*)$. This requires that q_n is not too large for any submarket n . Note that if $q_n = 1$, then the assumption is satisfied. To see this, note that when $q_n = 1$, V_n can be expressed as a weighted average of $U_n(y)$: $V_n = \int_{y_n} w_n(y) U_n(y) dF_n(y)$, where the quasi-weight $w_n(y)$ is given by

$$w_n(y) = \frac{r + \alpha[1 - G(\hat{\varepsilon}_n(y))]}{r + \alpha \int_{y_n} (1 - G(\hat{\varepsilon}_n(y))) dF_n(y)}. \quad (7)$$

It is easy to see that $w_n(y)$ is weakly increasing in y . Combining with the fact that $U_n(y)$ is increasing in y , we have $V_n < U_n(y_{n-1}^*)$.

We further impose the following assumption on the distribution of ε .

Assumption 3: $\frac{g(\varepsilon)}{r + \alpha(1 - G(\varepsilon))}$ is weakly increasing in ε .⁴¹

Lemma 7 *Suppose α increases to $\alpha' > \alpha$ while all the other parameter values of the model remain the same. In addition, suppose either $q_n \geq 1$ or q_n is close enough to 1 for each n , and the initial equilibrium segmentation \bar{S} does not change. Then for each n , (i) suppose Assumption 3 holds, $\hat{\varepsilon}^m \leq 1$, and $\frac{r + \alpha[1 - G(1)]}{g(1)} - \frac{\alpha^2 \gamma}{r} \geq 0$, then $V_n' > V_n$; (ii) $\Delta U_n(y)$ is increasing in y for all $y \in y_n$ with the property that $\hat{\varepsilon}_n(y) = \hat{\varepsilon}^m$; (iii) $\Delta U_{n+1}(y_n^*) > 0$; suppose Assumption 3 holds and $\frac{\partial V_n}{\partial \alpha} > \frac{\partial U_n(y_n^*)}{\partial \alpha}$, then $\Delta U_n(y_n^*) < \Delta U_{n+1}(y_n^*)$.*

The intuition for Lemma 7 is as follows. As men’s contact rate α increases, women’s contact rate αq_n increases as well, and this directly benefits women. On the other hand, an increase in α also makes men more choosy: men’s acceptance cutoff increases. This indirect effect tends to hurt women. However, since q_n is large enough (either close to 1 or larger than 1), the direct benefit outweighs the indirect loss.⁴² As a result, V_n must increase in α . This further implies that $\hat{\varepsilon}_n^w(y)$ increases.

⁴¹This assumption is a little stronger than the logconcavity of $g(\varepsilon)$, which is equivalent to $\frac{g(\varepsilon)}{1 - G(\varepsilon)}$ being weakly increasing in ε . Roughly speaking, this condition says that the density $g(\varepsilon)$ cannot decrease too fast. It is satisfied for a truncated normal distribution with a big enough variance and trivially satisfied for uniform distribution.

⁴²The fact that $q_n \geq 1$ means that the increase in women’s contact rate is bigger than that of men’s as α increases.

Part (iii) shows that a marginal type man y_n^* gains more from an increase in α by staying in the lower submarket $n + 1$ than staying in the higher submarket n . This result is driven by two opposite effects. First, relative to being in the lower submarket $n + 1$, in the higher submarket n a type y_n^* man's initial acceptance probability is lower and hence he can gain more in percentage terms. This effect favors the higher submarket n . The second effect, on the other hand, favors the lower submarket $n + 1$. Being the highest type in submarket $n + 1$, a type y_n^* man is not affected by an increase in V_{n+1} . However, he is the lowest type in submarket n , and thus an increase in V_n reduces his expected payoff. When V_n increases significantly in α (guaranteed by the condition $\frac{\partial V_n}{\partial \alpha} > \frac{\partial U_n(y_n^*)}{\partial \alpha}$, which says that women benefit more than the lowest type men in submarket n from an increase in α) and the induced increase in the acceptance cutoff significantly reduces the acceptance probability of type y_n^* men in submarket n (guaranteed by the assumption on the distribution), the second effect dominates the first one, and the marginal type men gain more from an increase in α by staying in the lower submarket.

Finally, we want to point out that the sufficient conditions specified in part (i) and (iii) are far from being necessary. In the appendix, we show that these two conditions are satisfied under some mild conditions.

4.2 With endogenous adjustment in segmentation

The endogenous adjustment of the segmentation is characterized in the following proposition.

Proposition 3 *Suppose α increases to α' while all the other parameter values of the model remain the same, and the conditions in Lemma 7 (parts (i) and (iii)) are satisfied. Then for each n : (i) y_n^* strictly increases; (ii) V_n strictly increases and $\widehat{\varepsilon}_n^w(y)$ strictly increases; (iii) the highest types of men (in submarket 1) are strictly better off; if the increase in α is not too large, then the highest types of men in submarket n , $1 < n \leq N$, are strictly better off, and the lowest types of men originally in submarket n , $1 \leq n < N$, are strictly better off as well.*

Proposition 3 generates several testable implications. First of all, it predicts that an Internet-induced increase in search efficiency makes the matching pattern more assortative horizontally, after taking into account the adjustment in the vertical sorting. That is, newlyweds are more compatible. The underlying reason is that with the higher search efficiency, people meet with potential partner more easily and thus become choosier.

Second, Proposition 3 shows that the sorting is also more assortative vertically. That is, there will be fewer men participating in higher submarkets and more men participating in lower submarkets. This could potentially reduce the intergenerational mobility. The underlying reason for this result is that, as women become choosier with higher search efficiency, in each

submarket low type men are affected more and they consequently switch to adjacent low submarkets.

The above two points also demonstrate that sorting in the vertical dimension and sorting in the horizontal dimension are closely related. When people become choosier about horizontal traits due to exogenous changes, such as an increase in search efficiency here, their utilities from participating in a given submarket alter and the sorting along the vertical dimension is affected. On the other hand, when the pattern of vertical sorting adjusts, women will change their standards on the horizontal attributes, as the vertical types of men available in the submarket now differ.

The empirical evidence in Lee (2009, 2016) are consistent with our predictions. She documents that, as online dating websites become more popular, people in Korea become more likely to marry a spouse with the same marital history and similar education levels. In other words, the Internet makes marriage more assortative.⁴³

Finally, regarding the expected payoffs, our model predicts that all women are unambiguously better off and the men who initially are among the highest and lowest types in a given submarket are also better off if the increase in α is small. The highest types are better off because they benefit directly from an increase in α , and an increase in women's value V_n does not affect their acceptance probabilities. The lowest types are also better off because, by switching to the adjacent lower market, they become the highest types in the new segmentation, and thus they get the direct benefit from an increase in α while avoiding being negatively affected by increases in women's values.⁴⁴

We use the following numerical example to illustrate the above results as well as to discover more quantitative patterns.

Example 2 *Suppose in the benchmark example α increases from 0.25 to 0.4, while all the other parameter values remain the same. The change in the equilibrium is illustrated in Figure 3 and Table 2.*

Table 2: How Equilibrium Variables Change as α Increases

⁴³It is hard to argue that marital history and education level are purely horizontal attributes or purely vertical attributes. Hitsch et al. (2010a) found that both men and women prefer mates with similar education level and marital history. This indicates that marital history and education level are horizontal attributes. But both education level and marital history could be correlated with income, thus both have components of vertical attributes as well.

⁴⁴For the middle types of men in each submarket (remaining in the same submarket), however, in general it is not analytically clear whether they are better off or worse off. This is because, although they benefit directly from an increase in α , they are negatively affected by women's becoming more choosy, and either effect can dominate.

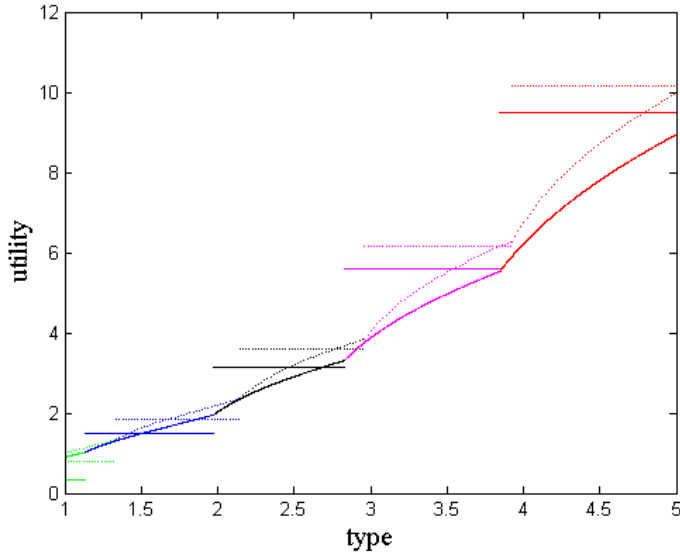


Figure 3: The Impacts of an Increase in α

	Submarket 1	Submarket 2	Submarket 3	Submarket 4	Submarket 5
$y_n^{*I} - y_n^*$	0.0782	0.1435	0.1740	0.2016	—
$V_n^{*I} - V_n^*$	0.6086	0.5291	0.4471	0.3655	0.4769
$q_n^{*I} - q_n^*$	-0.3018	-0.2307	-0.1276	-0.0221	0.2145

In Figure 3, the dotted curve and the dotted V -lines are associated with the new $\alpha = 0.4$. Figure 3 and Table 2 verify all predictions in Proposition 3.⁴⁵ Besides, it suggests some additional testable implications.

First of all, if we separate women into upper tail and lower tail according to the median type (which is type 4 women), an Internet-induced increase in search efficiency increases inequality among women in the upper tail (the increase in women's value is bigger in a higher submarket), but reduces inequality among women in the lower tail (the increase in V_5^* is bigger than that of V_4^* and V_3^* , because being the lowest submarket, submarket 5 is the only submarket that has a net gain in the number of participating men).

Second, an Internet-induced increase in search efficiency increases the overall inequality among men. Within each submarket (among the non-switching types), the gain in men's utility is increasing in men's type, as higher type men are affected less by an increase in women's value. Across submarkets, the general pattern is that the utility gains of men are higher in a higher

⁴⁵Note that this example does not satisfy all the assumptions of the analytical results. For instance, q_5^* is significantly below 1, $\hat{\varepsilon}_1(\bar{y}) > \hat{\varepsilon}^m$ and $\hat{\varepsilon}_2(y_1^*) > \hat{\varepsilon}^m$. The fact that all predictions still hold demonstrates the robustness of the results.

submarket.⁴⁶ This is because the induced adjustment in the endogenous segmentation reduces the number of men participating in higher submarkets significantly, but it reduces only slightly (or even increases) the number of those participating in lower submarkets (as reflected by the change in q_n in Table 2). This effect further increases men’s utilities in the highest submarket, but dampens the increases in the utilities of the men participating in the lower submarkets.

Finally, although search becomes more efficient, some people’s marrying speed may decrease (for example, the low types in submarkets 3 and 4). This is because, being the lowest types in their submarkets, these types of men are affected significantly by increases in women’s values, leading to significant increases in women’s acceptance cutoff for them. This indirect negative effect dominates the direct benefit of an increase in α on these men’s marrying speeds, and as a result the marrying speeds of these men decrease.

5 Changes in Women’s Population

5.1 An increase in the women to men sex ratio

Besides search efficiency, another important determinant of marriage market outcomes is the sex ratio. In this subsection, we investigate the impact of an increase in the women to men ratio. The sex ratio may change, for instance, when more couples prefer girls and sex selection is possible, or due to military mortality.

Proposition 4 *Suppose X_n increases for any n , while all the other parameter values of the model remain the same. Then for each n , (i) y_n^* ($n < N$) strictly decreases; (ii) V_n strictly decreases; and (iii) $U_n(y)$ strictly increases if in the original equilibrium $\hat{\varepsilon}_n(y) > \hat{\varepsilon}^m$ and stays unchanged otherwise.*

Proposition 4 shows that male scarcity results in a less assortative matching vertically: more men marry higher classes of women and fewer men marry lower classes. This is because, as the number of women in a submarket increases, an individual woman’s probability of meeting a potential mate decreases. Women then become less choosy and it attracts the inframarginal men to switch to the higher submarket.⁴⁷ This prediction fits existing empirical findings. For instance, Abramitzky et al. (2011) shows that, after the reduction of male population in WWI, men were less likely to marry women of lower social classes and more likely to marry women of higher social classes.

⁴⁶Note that the utility gains are not monotonically increasing in men’s types across submarkets: the types that switch to a lower submarket actually gain more than the non-switching types which are higher than the switching types but are among the lowest types in the submarket in question in the new equilibrium. However, the general pattern is still that the utility gains of men are higher in a higher submarket.

⁴⁷If the women to men ratio decreases, then the results are reversed.

Decreases in V_n also imply that newlyweds are less compatible. Another implication here is that while men can gain from population growth of women, the gain is not monotonic in men's type. Within a submarket, lower types gain more than higher types. Across submarkets, however, a higher type man can gain less than a lower type man if the former is among low types in a high submarket while the latter is among high types in a low submarket. The underlying reason for both results is that low types in a submarket benefit more from a reduction in women's payoff while high types have little gain.

5.2 An increase in X_n

In this subsection, we study how changes in X_n , the measure of n th class women, affect equilibrium.⁴⁸ Unlike a change in the sex ratio which affects the entire marriage market, this is a change confined in a single submarket. Yet we will show that it affects all submarkets. An immediate observation is that, as X_n changes, $\widehat{\varepsilon}^m$ will not change, as the equation that implicitly determines $\widehat{\varepsilon}^m$, (11), does not depend on X_n .

Proposition 5 *Suppose X_n increases to $X'_n > X_n$, while all the other parameter values of the model remain the same. Then the following results hold. (i) For $i \geq n$, y_i^* strictly decreases, V_i^* strictly decreases, and $U_i(y)$, $y \in y_i$, strictly increases. (ii) For $i \leq n - 1$, y_i^* strictly increases, V_i^* strictly decreases, and $U_i(y)$, $y \in y_i$, strictly increases, if $\widehat{\varepsilon}_j(y_{j-1}^*) > \widehat{\varepsilon}^m$ for all j , $i + 1 \leq j \leq n$; and y_i^* , V_i^* , and $U_i(y)$, $y \in y_i$, all remain the same if there is a j , $i + 1 \leq j \leq n$, such that $\widehat{\varepsilon}_j(y_{j-1}^*) = \widehat{\varepsilon}^m$.*

The intuition for Proposition 5 is as follows. When there are more class n women, if the initial equilibrium segmentation does not change, then the meeting rate of each class n woman decreases. As a result, their expected payoff V_n decreases and they become less choosy about men. Thus, men participating in this submarket get higher utilities. This attracts the inframarginal types (close to the types participating in submarket n) in adjacent submarkets to switch to submarket n , which further induces adjustments in other submarkets. This exercise illustrates how all submarkets are interlinked as discussed in the earlier section: a shock in the measure of women in one submarket will transmit to all submarkets through the endogenous adjustment in the market segmentation.

We also want to emphasize that the transmissions of a shock in submarket n are asymmetric in the upward and downward directions. When X_n increases, all lower class women (lower than n) and the corresponding types of men are definitely affected. However, higher class women and the corresponding types of men may not be. The underlying reason is that, for the lower

⁴⁸A particular type of women in a region may increase due to, for instance, the emerging of a new industry that attracts female immigrants of a particular social class.

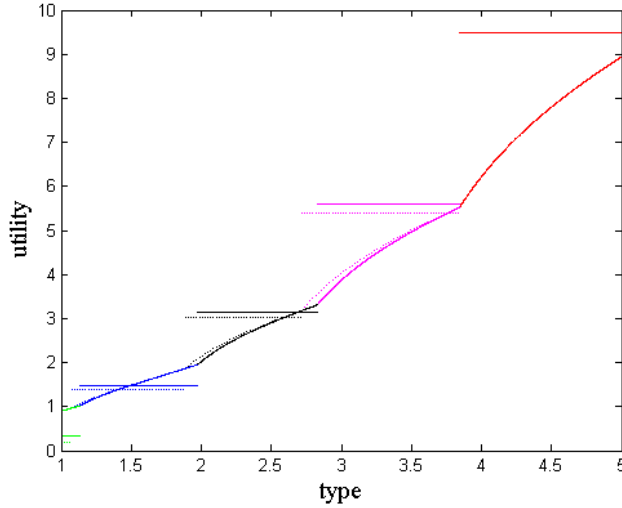


Figure 4: The Impacts of an Increase in X_2

marginal type men in any submarket j , women's acceptance cutoff always binds in equilibrium, thus a change in V_j will always affect this type of men's utility in submarket j . On the other hand, for the upper marginal type men in submarket j , men's acceptance cutoff might be binding, thus a decrease in V_j might not affect this type of men's utility at all. As a result, the upper bound y_{j-1}^* might not change, leaving all the higher class women and the corresponding men unaffected. Moreover, even if women's acceptance cutoff is binding for the upper marginal type men, they benefit a lot less from a reduction in V_j than the lower marginal type men. Therefore, the higher submarkets are affected much less. The following example demonstrates the last point.

Example 3 Suppose in the benchmark example X_2 increases from 0.1 to 0.13, while all the other parameter values remain the same. The change in the equilibrium is illustrated in Figure 4.

In Figure 4, the dotted curve and the dotted V -lines are associated with the new $X_2 = 0.13$. We can see that the highest submarket, submarket 1, is barely affected, and the lower submarkets are affected relatively more significantly.⁴⁹

⁴⁹The reduction in V_5^* is relatively more significant than those in submarkets 3 and 4. The reason is that, although submarkets 3 and 4 lose higher types of men to higher submarkets, they gain lower types of men from lower submarkets. In contrast, submarket 5 loses higher types of men without gaining lower types, as it is the lowest submarket.

6 Horizontal Match Value Becomes More Dispersed

As the standard of living increases, usually people will have more time to enjoy leisure and develop hobbies. As a consequence, in the modern and the post-modern world people's interests become more dispersed. This trend is more pronounced in countries that experience rapid income growth in a short period of time, such as Korea and China. In this section, we study the impacts of an increase in the dispersion of the horizontal match value on the marriage market outcome. In particular, we examine how this change in horizontal dimension would affect the marriage sorting vertically.

For simplicity, we assume that the match value ε is uniformly distributed on $[1 - \gamma, 1 + \gamma]$, and use an increase in γ to capture the idea that the match value ε becomes more dispersed.⁵⁰ In some sense, an increase in γ means that the horizontal match fitness becomes relatively more important. Again we will conduct the analysis in two steps: in the first step we hypothetically fix the initial equilibrium segmentation, and in the second step we study how the segmentation will endogenously adjust.

6.1 Fixed segmentation

Lemma 8 *Suppose γ increases to $\gamma' > \gamma$, while all the other parameter values of the model remain the same. Assume the initial equilibrium segmentation \bar{S} does not change. Then, (i) $\widehat{\varepsilon}^m$ strictly increases; (ii) for each n , $U_n(y)$ strictly increases for any y with $\widehat{\varepsilon}_n(y) = \widehat{\varepsilon}^m$; (iii) for each n , V_n strictly increases if q_n is close to 1 and $2\widehat{\varepsilon}^m < (1 + \gamma) + \frac{2r\gamma}{\alpha}$; (iv) for each n , $\frac{\partial U_n(y_n^*)}{\partial \gamma} > \frac{\partial U_{n+1}(y_n^*)}{\partial \gamma}$ if and only if $\frac{\partial V_n}{\partial \gamma} < \widehat{x} \frac{\partial U_n(y_n^*)}{\partial \gamma}$, where*

$$\widehat{x} = \frac{\theta_n}{\theta_{n+1}} \left\{ 1 + \left(\frac{\theta_n}{\theta_{n+1}} - 1 \right) \frac{(1 + \gamma) \left[1 + \frac{2r}{\alpha} \frac{\gamma + \widehat{\varepsilon}^m}{1 + \gamma - (1 + \frac{2r}{\alpha}) \widehat{\varepsilon}^m} \right] - \widehat{\varepsilon}_n(y_n^*)}{\frac{\theta_n}{\theta_{n+1}} \widehat{\varepsilon}_n(y_n^*) - \widehat{\varepsilon}^m} \right\}.^{51} \quad (8)$$

The intuition for parts (i)-(iii) of Lemma 8 is as follows. As the horizontal match value becomes more dispersed, men and women have higher probabilities of meeting someone with a higher horizontal match value. This clearly benefits men of high types (within a submarket) whose acceptance threshold is effective. Their equilibrium payoffs are higher and they become pickier. Just like an increase in α , an increase in γ has two opposite effects on women. The direct effect is that it benefits women, as they now have higher probabilities of meeting men with higher horizontal match values. However, now higher types of men are pickier, which restricts women's choices, and this indirect effect tends to hurt women. However, when the

⁵⁰Essentially, an increase in γ leads to a mean preserving spread of the distribution of ε .

⁵¹How big is the critical value of \widehat{x} ? From equation (8), it can be easily verified that $\widehat{x} > \frac{\theta_n}{\theta_{n+1}} > 1$, or \widehat{x} is relatively big. As a result, it is relatively easy for $\frac{\partial V_n}{\partial \gamma} < \widehat{x} \frac{\partial U_n(y_n^*)}{\partial \gamma}$ to hold.

number of women and men is similar, the direct effect dominates, and overall women benefit from an increase in γ .⁵²

Part (iv) of Lemma 8 also shows that when the increase in V_n^* is small compared to the increase in $U_n(y_n^*)$, then the marginal type men benefit more from an increase in the horizontal match value by staying in the higher submarket n . There are three effects behind this result. First, since type y_n^* is the highest type in submarket $n + 1$, an increase in V_{n+1} induced by an increase in γ does not affect $U_{n+1}(y_n^*)$. However, an increase in V_n will reduce type y_n^* 's probability of being accepted in submarket n , which dampens $U_n(y_n^*)$. This effect favors submarket $n + 1$. Second, an increase in γ means that the marginal type has a higher probability of finding larger horizontal match values. In percentage terms, this effect is stronger in submarket n as this type's initial equilibrium probability of being accepted is smaller in submarket n , which means that the marginal type y_n^* gains more by staying in the higher submarket n . Notice that these two effects are also present when α increases. The third effect is new. It stems from the supermodularity of men's payoff in women's type and the horizontal match value. Then by staying in the higher submarket n , the marginal type men can gain more. Overall, if $\frac{\partial V_n}{\partial \gamma}$ is relatively not too big so that the first effect is not too strong, then the second and third effects dominate. Recall that with an increase in α , we reach the opposite conclusion: the marginal type men benefit more from staying in the lower submarket. The underlying reason is that the third effect is absent when α changes.

6.2 With endogenous adjustment in segmentation

We use superscripts “ a ” and “ m ” to indicate variables after γ increases to γ' when the segmentation is held constant and when the segmentation adjusts endogenously, respectively. The analysis in the previous subsection shows that the comparison between $U_n^a(y_n^*)$ and $U_{n+1}^a(y_n^*)$ can go either way depending on whether $\frac{\partial V_n}{\partial \gamma}$ is larger than $\hat{x} \frac{\partial U_n(y_n^*)}{\partial \gamma}$. To state cleaner analytical results, we focus on the cases where $U_n^a(y_n^*)$ is always larger than $U_{n+1}^a(y_n^*)$ in any submarkets, as the previous analysis suggests that this is more likely to happen.

Proposition 6 *Suppose γ increases to γ' , while all the other parameter values of the model remain the same. If $U_n^a(y_n^*) > U_{n+1}^a(y_n^*)$ for all n , then for each n : (i) y_n^* strictly decreases; (ii) $V_1' > V_1$ if q_1 is close enough to 1 and $2\hat{\varepsilon}^m < (1 + \gamma) + \frac{2r\gamma}{\alpha}$; (iii) the highest types of men (initially in submarket 1) with acceptance threshold $\hat{\varepsilon}_n(y) = \hat{\varepsilon}^m$ are strictly better off; if the change in γ is not too large, then the highest types of men initially in any submarket whose acceptance threshold is still $\hat{\varepsilon}_n(y) = \hat{\varepsilon}^m$ are strictly better off, and the lowest types of men who are initially in submarket n , $1 \leq n < N$, are strictly better off as well.*

⁵²The sufficient condition $2\hat{\varepsilon}^m < (1 + \gamma) + \frac{2r\gamma}{\alpha}$ is satisfied if r/α is not too small, the same condition as in the case of an increase in α . This sufficient condition is far from being necessary.

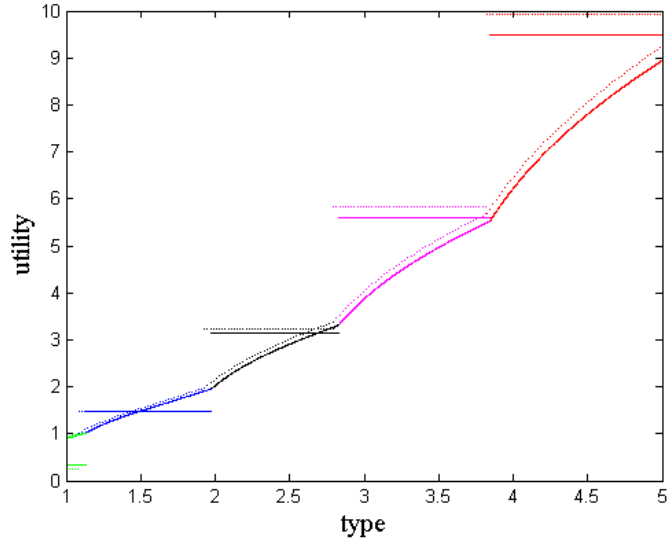


Figure 5: The Impacts of an Increase in γ

The main testable implication of Proposition 6 is that as the horizontal match value becomes more dispersed, the matching pattern becomes less assortative in the vertical dimension.⁵³ We want to point out that this result is the opposite to the change in the matching pattern when α increases, under which the marriage pattern becomes more assortative. The main reason, as mentioned in the last subsection, is that an increase in γ , due to the complementarity between the vertical types and the horizontal match value, makes the higher submarket relatively more attractive. This effect, which is absent when α increases, dominates the negative effects and attracts inframarginal men to move to the adjacent higher submarket.

The following numerical example shows some additional testable implications concerning the utility inequality in the marriage market.

Example 4 *Suppose in the benchmark example γ increases from 0.5 to 0.6, while all the other parameter values remain the same.⁵⁴ The change in the equilibrium is illustrated in Figure 5 and Table 3.*

Table 3: How Equilibrium Variables Change as γ Increases

	Submarket 1	Submarket 2	Submarket 3	Submarket 4	Submarket 5
$V_n^{*f} - V_n^*$	0.4866	0.2389	0.0933	0.0043	-0.1201
$q_n^{*f} - q_n^*$	0.0797	0.0592	0.0315	0.0037	-0.0528

⁵³This result is not obtained in Herrenbrueck et al. (2016).

⁵⁴Again, this example does not satisfy all the assumptions of the analytical results.

In Figure 5, the dotted curve and the dotted V -lines are associated with the new $\gamma = 0.6$. Figure 5 and Table 3 indicate the following empirical implications of an increase in γ . First, an increase in the dispersion of the horizontal match value enlarges the inequality among women. We can see that the increase in women's value increases in women's type. The underlying reason for this pattern is that more men switch to higher submarkets, which benefits women in higher submarkets a lot by increasing their contact rate significantly. In contrast, this only slightly benefits (or even hurts) women in lower submarkets by barely increasing (or decreasing) their contact rate (as shown in the changes in q_n in Table 3).

Second, the inequality among men also increases. Within each submarket, the increase in men's utility is increasing in men's type, because higher type men are affected less by an increase in women's value. The same pattern also holds across submarkets. For instance, men in submarkets 4 and 5 gain very little, while men in submarkets 1 and 2 gain significantly.

Finally, comparing men's and women's gains across different types, we see that the induced increase in inequality is more significant among women than among men. This is due to the endogenous adjustment in segmentation. With men switching to higher submarkets, women's gains in higher submarkets are increased further, while men's gains in higher submarkets are dampened.

7 Horizontal Clubs

As mentioned earlier, empirical evidence indicates that couples are more compatible if they share the same core value/interests. The Internet not only increases the search efficiency (the contact rate α), but also enables men and women to create horizontal clubs in which they meet people who shares the same core value/interests. For instance, there is an online dating site called Farmers, which attracts men and women who are interested in becoming farmers. That is, the Internet also allows people to target specific horizontal attributes. This targeting will naturally increase the horizontal match value between men and women participating in the same horizontal club, as they share some common core interests.

To study how this possibility of horizontal targeting affects the equilibrium marriage pattern, we modify our baseline model in a simple way. Specifically, assume that both men and women have $T \geq 2$ possible horizontal types, and each type is equally likely. Moreover, each man's or woman's horizontal type is independent of his/her vertical type. If a man or woman matches with a partner of the same horizontal type, then the horizontal match value is randomly drawn from distribution \tilde{G} , which is a truncation of G after restricting the support to $[\underline{x}, 1 + \gamma]$, $\underline{x} > 1 - \gamma$. On the other hand, if the two partners are of different horizontal types, then the horizontal match value is randomly drawn from \check{G} . To make the comparison

reasonable, we assume that $\frac{1}{T}\tilde{G} + \frac{T-1}{T}\check{G} = G$ for any ε in the support $[1 - \gamma, 1 + \gamma]$.⁵⁵ Assume further that there are T horizontal clubs in each submarket and men can also target horizontal clubs. That is, in total there are $T \times N$ submarkets, or each vertical submarket n consists of T distinctive horizontal clubs, $\{n_1, \dots, n_t, \dots, n_T\}$. Once two agents decide to marry and leave the market, they are replaced by clones with the same vertical and horizontal types.

Since the horizontal clubs are symmetric, men's horizontal types are symmetric, and \tilde{G} first-order stochastically dominates G , each type t men will always target the corresponding type t women clubs. Given this feature, we can analyze each type t agents separately, and the equilibrium among each type t agents will be exactly the same. For this reason, we only focus on one particular horizontal club in the following analysis. Now compare the baseline model and the model with horizontal targeting. Specifically, in the process we fix α and γ , but only change the distribution of ε from G to \tilde{G} . Let $k \equiv \frac{1}{1-G(\underline{x})} > 1$. Note that for $\varepsilon \in [\underline{x}, 1 + \gamma]$, $\tilde{g}(\varepsilon) = kg(\varepsilon)$.

In addition, we only focus on the range of parameter values under which men's acceptance threshold is interior with horizontal targeting. Following a similar analysis as Lemma 1, men's acceptance threshold $\hat{\varepsilon}^{m'}$ under \tilde{G} is implicitly determined by the following condition when it is interior:

$$-r\hat{\varepsilon}^{m'} + \alpha \int_{\hat{\varepsilon}^{m'}}^{1+\gamma} [\varepsilon - \hat{\varepsilon}^{m'}] d\tilde{G}(\varepsilon) = 0. \quad (9)$$

Thus, the following assumption guarantees that $\hat{\varepsilon}^{m'}$ is strictly greater than \underline{x} .

Assumption 1A: $\int_{\underline{x}}^{1+\gamma} \varepsilon d\tilde{G}(\varepsilon) > (1 + \frac{r}{\alpha})\underline{x}$.

In particular, this requires that α should not be too small. It is also easy to see that $\hat{\varepsilon}^{m'}$ equals $\hat{\varepsilon}^m$ in the baseline model but with α changed to $k\alpha$, as equation (9) can be rewritten as

$$-r\hat{\varepsilon}^m + k\alpha \int_{\hat{\varepsilon}^m}^{1+\gamma} [\varepsilon - \hat{\varepsilon}^m] dG(\varepsilon) = 0,$$

using the fact that $\tilde{g}(\varepsilon) = kg(\varepsilon)$ for $\varepsilon \in [\underline{x}, 1 + \gamma]$. Then $\hat{\varepsilon}^{m'} > \underline{x}$ implies that $\hat{\varepsilon}^m$ in the baseline model with $k\alpha$ is also greater than \underline{x} .

The following proposition shows that adding horizontal clubs is equivalent to an increase in α .

Proposition 7 *Adding horizontal clubs while keeping all the other parameter values of the model unchanged, is equivalent to an increase in α in the baseline model.*

⁵⁵Given G , \tilde{G} , and T , there is a \check{G} satisfying $\frac{1}{T}\tilde{G} + \frac{T-1}{T}\check{G} = G$. The details of \check{G} are not essential for our results.

Proposition 7 suggests that we can use the comparative statics results regarding α to compare the equilibrium outcomes with and without horizontal clubs. In particular, adding horizontal clubs makes the marriage pattern more assortative in the vertical dimension, benefits women in all submarkets, etc.⁵⁶ Moreover, the two effects of the Internet on the marriage market, through an increase in α and enabling horizontal targeting, work exactly in the same direction.

The analysis above only covered the case that the horizontal types are symmetric: both genders have the same distribution and the distribution is uniform. When the distributions of the horizontal types are different across men and women, with horizontal targeting an interesting feature will emerge in equilibrium: some vertical types of men of a relatively more popular horizontal type will participate in women clubs different from their corresponding (horizontal) type. In other words, in equilibrium horizontal targeting is not perfect but hybrid, though perfect targeting is technologically feasible.⁵⁷

To simplify analysis, we assume that there are only two vertical types of women, θ_1 and θ_2 . Moreover, for both male and female there are only two horizontal types, L and R . We use λ_w and λ_m , $\lambda_w \neq \lambda_m$, to denote the proportions of type L women and men, respectively. In particular, $\lambda_m > \lambda_w$; that is, type L men are relatively more mainstream than type L women. If a pair of man and woman of the same horizontal type meet, the match value ε is randomly drawn from distribution G_g (the subscript g is short for “good”). If a pair of man and woman of the opposite horizontal types meet, ε is randomly drawn from distribution G_b (the subscript b is short for “bad”). G_g first-order stochastically dominates G_b . Note that with horizontal targeting there are four clubs for women in total, which are denoted as $L1$, $L2$, $R1$, and $R2$ (for instance, club $L1$ is the club of women with horizontal type L and vertical type θ_1). Women’s expected payoffs in the four clubs are denoted as V_{L1} , V_{L2} , V_{R1} , and V_{R2} , respectively. With horizontal targeting each man is free to choose beforehand which club to target their search.

To proceed, first consider the case of perfect horizontal targeting: type L men only search in the L clubs, and type R men only search in the R clubs. In this case, the vertical segmentation of type L men and that of type R men are different. To see this, denote \hat{y}_L and \hat{y}_R as the marginal (vertical) types of men for the L type and the R type, respectively. Suppose the vertical cutoff types are the same across horizontal clubs, $\hat{y}_L = \hat{y}_R \equiv \hat{y}$. This implies that, since $\lambda_m > \lambda_w$, the men to women ratio in club $L1$ is higher than that of club $R1$: $q_{L1} = \frac{\lambda_m}{\lambda_w} \frac{F(\hat{y})}{Y_1} > \frac{1-\lambda_m}{1-\lambda_w} \frac{F(\hat{y})}{Y_1} = q_{R1}$. Similarly, we can show that $q_{L2} > q_{R2}$. Since $q_{L1} > q_{R1}$,

⁵⁶This result should be interpreted with caution, because in the real world club formation is not unconstrained. More specifically, forming a particular horizontal club might change the distribution of vertical types. For instance, in a farming club agents’ income distribution is very likely to be more homogeneous than that of the general population.

⁵⁷This extension is suggested by a referee.

women in club $L1$ gets a higher value than women in club $R1$, $V_{L1} > V_{R1}$. As a result, women in club $L1$ are pickier than women in club $R1$ in terms of the acceptance cutoff: $\widehat{\varepsilon}_{L1}^w(y) > \widehat{\varepsilon}_{R1}^w(y)$ for any y . This means that $\widehat{y}_L = \widehat{y}_R \equiv \widehat{y}$ cannot be an equilibrium.⁵⁸ In order to achieve equilibrium in both horizontal clubs, we must have $\widehat{y}_L > \widehat{y}_R$. This is because, compared to the middle types of R men with respect to clubs $R1$ and $R2$, for middle types of L men club $L2$ is relatively more attractive than club $L1$. We can further show that the following properties hold with perfect targeting: $V_{L1} > V_{R1}$, and $\widehat{\varepsilon}_{L1}^w(y) > \widehat{\varepsilon}_{R1}^w(y)$ for any y .⁵⁹

However, perfect targeting may not be an equilibrium. This is because type L men with vertical types around \widehat{y}_L might have an incentive to target club $R1$ where women are less picky. To check these men's incentives, consider a type L man of vertical type \widehat{y}_L . Note that this type is indifferent between club $L1$ and $L2$. Does he have an incentive to deviate to club $R1$? The difference between his deviation payoff in club $R1$, $U_{R1}(\widehat{y}_L)$, and his payoff in club $L1$, $U_{L1}(\widehat{y}_L)$, can be written as

$$U_{R1}(\widehat{y}_L) - U_{L1}(\widehat{y}_L) = \alpha\theta_1\widehat{y}_L \left[\frac{\int_{\widehat{\varepsilon}_{R1}^w(\widehat{y}_L)}^{1+\gamma} \varepsilon dG_b(\varepsilon)}{r + \alpha[1 - G_b(\widehat{\varepsilon}_{R1}^w(\widehat{y}_L))]} - \frac{\int_{\widehat{\varepsilon}_{L1}^w(\widehat{y}_L)}^{1+\gamma} \varepsilon dG_g(\varepsilon)}{r + \alpha[1 - G_g(\widehat{\varepsilon}_{L1}^w(\widehat{y}_L))]} \right].$$

Thus, he has an incentive to deviate if

$$\frac{\int_{\widehat{\varepsilon}_{R1}^w(\widehat{y}_L)}^{1+\gamma} \varepsilon dG_b(\varepsilon)}{r + \alpha[1 - G_b(\widehat{\varepsilon}_{R1}^w(\widehat{y}_L))]} - \frac{\int_{\widehat{\varepsilon}_{L1}^w(\widehat{y}_L)}^{1+\gamma} \varepsilon dG_g(\varepsilon)}{r + \alpha[1 - G_g(\widehat{\varepsilon}_{L1}^w(\widehat{y}_L))]} > 0. \quad (10)$$

Note that condition (10) is satisfied if $\widehat{\varepsilon}_{L1}^w(\widehat{y}_L)$ is significantly higher than $\widehat{\varepsilon}_{R1}^w(\widehat{y}_L)$, which is likely if λ_m is significantly higher than λ_w .⁶⁰

Now suppose that condition (10) is satisfied for type \widehat{y}_L of L men. By continuity, L men with types around \widehat{y}_L will also deviate to participating in club $R1$. For those types above \widehat{y}_L they tradeoff a “worse” distribution G_b with a lower acceptance cutoff $\widehat{\varepsilon}_{R1}^w(y)$. For those types below \widehat{y}_L (initially targeting club $L2$) they tradeoff a “worse” distribution G_b and a higher acceptance cutoff $\widehat{\varepsilon}_{R1}^w(y)$ with a higher vertical type of women θ_1 . This adjustment reduces V_{L1} and makes women in club $L1$ less choosy, and it increases V_{R1} and makes women in club $R1$ more choosy, which helps to restore equilibrium.

⁵⁸Specifically, the condition that type \widehat{y} is the marginal type in L clubs and the condition that type \widehat{y} is also the marginal type in R clubs cannot be satisfied at the same time.

⁵⁹To show the properties, suppose $V_{L1} \leq V_{R1}$. Denote y_L as a type L man with vertical type y_L . Since $V_{L1} \leq V_{R1}$, we have $\widehat{\varepsilon}_{L1}^w(y) \leq \widehat{\varepsilon}_{R1}^w(y)$ for any y . In particular, $\widehat{\varepsilon}_{L1}^w(y_L = \widehat{y}_R) \leq \widehat{\varepsilon}_{R1}^w(\widehat{y}_R)$, which means that $U_{L1}(y_L = \widehat{y}_R) \geq U_{R1}(\widehat{y}_R)$. Due to $\widehat{y}_L > \widehat{y}_R$ and $\lambda_m > \lambda_w$, $q_{L2} > q_{R2}$. This implies that $V_{L2} > V_{R2}$, $\widehat{\varepsilon}_{L2}^w(y_L = \widehat{y}_R) \geq \widehat{\varepsilon}_{R2}^w(\widehat{y}_R)$, and $U_{L2}(y_L = \widehat{y}_R) \leq U_{R2}(\widehat{y}_R)$. Since \widehat{y}_R is the indifferent vertical type for type R men, $U_{R1}(\widehat{y}_R) = U_{R2}(\widehat{y}_R)$. Taken together, we have $U_{L1}(y_L = \widehat{y}_R) \geq U_{L2}(y_L = \widehat{y}_R)$; that is, type $y_L = \widehat{y}_R$ prefers club $L1$ to club $L2$. This contradicts the fact that type $\widehat{y}_L (> \widehat{y}_R)$ is indifferent between clubs $L1$ and $L2$. Therefore, we must have $V_{L1} > V_{R1}$, which further implies that $\widehat{\varepsilon}_{L1}^w(y) > \widehat{\varepsilon}_{R1}^w(y)$ for any y .

⁶⁰To see this, consider the limiting case that $\lambda_w \rightarrow 0$ but $\lambda_m > 0$. In this case, with perfect targeting q_{L1} goes to infinity and $\widehat{\varepsilon}_{L1}^w(\widehat{y}_L) \rightarrow 1 + \gamma$. As a result, the second term in the LHS of (10) converges to 0 while the first term is strictly positive, which implies that condition (10) is satisfied.

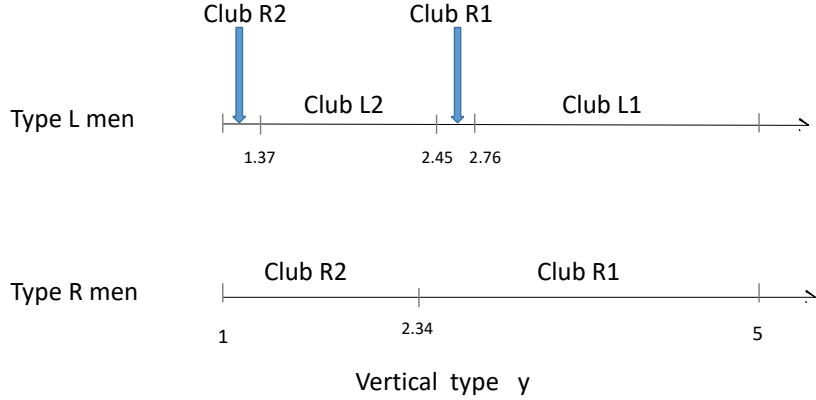


Figure 6: The Equilibrium Segmentation/Targeting Pattern

When condition (10) is satisfied for type \hat{y}_L of L men, a similar condition is also very likely to hold for the lowest types of L men (between clubs $L2$ and $R2$), as they also tradeoff a “worse” distribution G_b with a lower acceptance cutoff $\hat{\varepsilon}_{R2}^w(y)$. If this is the case, then the lowest types of L men will participate in club $R2$ instead of club $L2$. In equilibrium, the segmentation of type L men is as follows. Types above \hat{y}_L^{***} participate in club $L1$, types within $[\hat{y}_L^*, \hat{y}_L^{***}]$ participate in club $R1$ ($\hat{y}_L^* < \hat{y}_L < \hat{y}_L^{***}$), types within $[\hat{y}_L^*, \hat{y}_L^{**}]$ participate in club $L2$, and types below \hat{y}_L^* participate in club $R2$. The segmentation of type R men is characterized by \hat{y}_R^* ($\hat{y}_R^* > \hat{y}_R$), with types above \hat{y}_R^* participating in club $R1$ and those below \hat{y}_R^* participating in club $R2$.

The above analysis is illustrated in the following example.

Example 5 Suppose $\lambda_m = 2/3$, $\lambda_w = 1/2$, $\theta_1 = 2$, $\theta_2 = 1$, $X_1 = 0.2$, $X_2 = 0.8$, $r = 0.05$, $\alpha = 0.1$, and men’s type distribution is the same as in Example 1. G_g is uniform on $[0.5, 1.5]$, and G_b has two portions: in the first portion it is uniform on $[0.5, 1]$ with density 1.2, and in the second portion it is uniform on $[1, 1.5]$ with density 0.8. Under perfect horizontal targeting, in the L clubs $\hat{y}_L = 2.61$, $V_{L1} = 6.99$, and $V_{L2} = 1.17$; in the R clubs $\hat{y}_R = 2.17$, $V_{R1} = 5.83$, and $V_{R2} = 0.59$. For the marginal type 2.61 of the L type men, condition (10) is satisfied (deviating to club $R1$ can gain about 0.81 in expected payoff). Moreover, a condition similar to condition (10) is also satisfied for the type L men with the lowest vertical type 1. The equilibrium segmentation/targeting pattern is illustrated in Figure 6, with type L men exhibiting hybrid targeting.

8 Conclusion and Discussion

This paper, within a search/matching framework, studies the equilibrium marriage pattern with vertically differentiated men and women. Our model has two distinctive features. First, men and women are also horizontally differentiated. Second, search is targeted: men can choose beforehand in which submarket to participate, but search is random within each submarket. We show that there is a unique market equilibrium. Men (who actively search) are endogenously segmented into different submarkets, and the matching pattern is weakly positive assortative. In equilibrium, all the submarkets are indirectly linked. Within each submarket, higher types of men are faster to marry than lower types. This is because the presence of horizontal differentiation means that different types of men will be accepted by women with different probabilities, which in some sense serves a shadow price.

An Internet-induced increase in search efficiency causes all the threshold types of men to increase, leading to overall more assortative matching in the vertical dimension. In the horizontal dimension, all the equilibrium acceptance cutoffs increase, which implies more assortative matching in the horizontal dimension as well. All women are always better off, but not all men are better off. Our simulation shows that in the upper tail women's inequality (in the marriage market) increases, but in the lower tail women's inequality decreases, while men's inequality in the marriage market in general increases. We also show that an Internet-induced horizontal targeting has similar qualitative impacts on the marriage market as an Internet-induced increase in search efficiency.

When the measure of a specific type of women increases, the corresponding submarkets attracts more men. Moreover, the transmissions of the shock are asymmetric. In the downward direction (the submarkets lower than the one where the original shock occurs), all the threshold types of men decrease. However, in the upward direction, the shock transmission can stop at any submarket.

Finally, as the dispersion of the horizontal match fitness increases, the equilibrium cutoff types of men are very likely to decrease, leading to overall less assortative matching. High types of women benefit from a more dispersed match value, but this may not be the case for low type women. For men, not all of them benefit as the dispersion of the horizontal match fitness increases. Our simulation results indicate that a more dispersed match value increases women's inequality as well as men's inequality, with the increase in women's inequality being more significant than that of men's.

In the rest of this section, we discuss two simplifying assumptions we have made.

Urn ball matching technology More generally, it is reasonable to think that the contact rate of men in a submarket is also decreasing in the men/women ratio in that submarket. This

direct search congestion is the key driving force in the directed search models in the labor market (such as Shimer, 2005). As mentioned earlier, in our model there are already indirect externalities imposed by men on each other within the same submarket. Introducing the direct search congestion would not qualitatively affect the main results of our model. Moreover, in the existing literature of search/matching models in the marriage market, it is commonly assumed that men and women have contact rates that are independent of the men/women ratio.

The clone assumption Among the marriage search/matching models, the clone assumption is standard, with BC being an exception. Like BC, we can introduce exogenous separation for existing marriages or exogenous entry of new mates and study the steady-state equilibrium. It is straightforward to see that the main equilibrium features mentioned in Section 3.3 remain valid. However, without the clone assumption, the analysis become significantly more complicated. The difficulty is that within each submarket the higher types of men are faster to marry. Moreover, different types of women have different marrying speeds as well. Quantitatively, we conjecture that in the steady-state equilibrium the utility differences between the types who are faster to marry and those who are slower to marry will be less pronounced than in the model with clones, as the former will be fewer and the latter will be more in the steady-state equilibrium. We will leave a complete analysis of the steady-state equilibrium for future research.

Appendix

Proof of Lemma 1.

Proof. First suppose that type n women always accept a type y man (we can drop the constraint $\hat{\varepsilon}_n^m(y) \geq \hat{\varepsilon}_n^w(y)$) and this man adopts a reservation strategy of accepting a type n woman if and only if $\varepsilon \geq \hat{\varepsilon}$. By (1), we can solve $U_n(y)$ as a function of $\hat{\varepsilon}$. Now take derivative of $U_n(y)$ with respect to $\hat{\varepsilon}$, we get

$$\frac{\partial U_n(y)}{\partial \hat{\varepsilon}} \propto -r\hat{\varepsilon} + \alpha \int_{\hat{\varepsilon}}^{1+\gamma} (\varepsilon - \hat{\varepsilon}) dG(\varepsilon) \equiv \Gamma(\hat{\varepsilon}). \quad (11)$$

It can be shown that $\Gamma(\hat{\varepsilon})$ is monotonically decreasing in $\hat{\varepsilon}$:

$$\frac{\partial \Gamma(\hat{\varepsilon})}{\partial \hat{\varepsilon}} \propto -r - \alpha(1 - G(\hat{\varepsilon})) < 0.$$

Moreover, $\Gamma(1 + \gamma) < 0$. When Assumption 1 is satisfied, $\Gamma(1 - \gamma) > 0$. Thus there is a unique $\hat{\varepsilon}^m \in (1 - \gamma, 1 + \gamma)$ such that $\Gamma(\hat{\varepsilon}^m) = 0$. Otherwise, $\Gamma(\hat{\varepsilon}) < 0$ for any $\hat{\varepsilon} \in (1 - \gamma, 1 + \gamma]$ and therefore $\hat{\varepsilon}^m = 1 - \gamma$. In both scenarios, all men participating in any submarket have the

same optimal reservation match value: accept any woman with $\varepsilon \geq \widehat{\varepsilon}^m$, and reject all women with $\varepsilon \leq \widehat{\varepsilon}^m$.

Note that the cutoff that ensures matching between a type y man and a type n woman is $\widehat{\varepsilon}_n(y) = \max\{\widehat{\varepsilon}^m, \widehat{\varepsilon}_n^w(y)\}$. If $\widehat{\varepsilon}_n^w(y) \leq \widehat{\varepsilon}^m$, a type y man's optimal cutoff is still $\widehat{\varepsilon}^m$, same as the previous analysis. If $\widehat{\varepsilon}_n^w(y) > \widehat{\varepsilon}^m$, type y men are willing to match type n women with $\varepsilon \geq \widehat{\varepsilon}_n^w(y)$, as $\Gamma(\widehat{\varepsilon}_n^w(y)) < 0$ by the previous analysis. In either case, we can write the optimal cutoff of any type of men as $\widehat{\varepsilon}^m$.

To see that $\widehat{\varepsilon}^m$ strictly increases in α when Assumption 1 holds, take derivative of Γ with respect to α and plug in the first order condition. It is easy to see that $\frac{\partial \Gamma(\widehat{\varepsilon})}{\partial \alpha} \big|_{\widehat{\varepsilon}=\widehat{\varepsilon}^m} > 0$. We already know that $\frac{\partial \Gamma(\widehat{\varepsilon})}{\partial \widehat{\varepsilon}} \big|_{\widehat{\varepsilon}=\widehat{\varepsilon}^m} < 0$. By the implicit function theorem, the claim is proved. ■

Proof of Lemma 2.

Proof. Part (i). Suppose there exists some $n > 1$ such that $\frac{V_{n-1}}{\theta_{n-1}} < \frac{V_n}{\theta_n}$. Since $\widehat{\varepsilon}_n(y) = \max\{\widehat{\varepsilon}^m, \frac{V_n}{\theta_n y}\}$, this implies $\widehat{\varepsilon}_{n-1}(y) \leq \widehat{\varepsilon}_n(y)$ for any y . In other words, women in submarket $n-1$ are less picky. As $\theta_{n-1} > \theta_n$, by (1), $\widehat{\varepsilon}_{n-1}(y) \leq \widehat{\varepsilon}_n(y)$ implies that $U_{n-1}(y) > U_n(y)$ for any y with $\widehat{\varepsilon}_{n-1}(y) < 1 + \gamma$ (or $y > \frac{V_{n-1}}{\theta_{n-1}(1+\gamma)}$). Thus these men will all participate in submarket $n-1$ and will not participate in submarket n . For any y such that $\widehat{\varepsilon}_{n-1}(y) \geq 1 + \gamma$ (or $y \leq \frac{V_{n-1}}{\theta_{n-1}(1+\gamma)}$), they will never marry a type n woman as they will always be rejected. As a result, $V_n = 0$. However, $V_{n-1} \geq 0$. This contradicts the initial presumption that $\frac{V_{n-1}}{\theta_{n-1}} < \frac{V_n}{\theta_n}$.

Part (ii). It directly follows part (i). ■

Proof of Lemma 3.

Proof. Suppose in a market equilibrium the scenario described in this statement occurs. It would mean that $U_{n'}(y) \geq U_n(y)$, $U_n(y') \geq U_{n'}(y')$, and $U_n(y) > 0$. Since $U_n(\cdot)$ is increasing, we have $U_{n'}(y) > 0$, $U_n(y') > 0$, and $U_{n'}(y') > 0$. These imply that all the relevant acceptance thresholds are strictly less than $1 + \gamma$. Define

$$\frac{\int_{\widehat{\varepsilon}}^{1+\gamma} \varepsilon dG(\varepsilon)}{r + \alpha(1 - G(\widehat{\varepsilon}))} \equiv H(\widehat{\varepsilon}).$$

Then, $U_n(y) = \alpha \theta_n y H(\widehat{\varepsilon}_n(y))$. Note that $H(\widehat{\varepsilon})$ is strictly decreasing in $\widehat{\varepsilon}$ for $\widehat{\varepsilon} \geq \widehat{\varepsilon}^m$ (proof of Lemma 1). Since $\widehat{\varepsilon}_n(y)$ is weakly decreasing in n for any y , we have the following three cases to consider.

Case (1): $\widehat{\varepsilon}_n(y) = \widehat{\varepsilon}_{n'}(y) = \widehat{\varepsilon}^m$.

Since $\widehat{\varepsilon}_n(\cdot)$ is decreasing in y , we have $\widehat{\varepsilon}_n(y') = \widehat{\varepsilon}_{n'}(y') = \widehat{\varepsilon}^m$. Now,

$$U_{n'}(y') - U_n(y') = \alpha y (\theta_{n'} - \theta_n) H(\widehat{\varepsilon}^m) > 0,$$

is a contradiction.

Case (2): $\widehat{\varepsilon}_n(y) = \widehat{\varepsilon}^m$ and $\widehat{\varepsilon}_{n'}(y) > \widehat{\varepsilon}^m$.

Since $\widehat{\varepsilon}_n(\cdot)$ is decreasing in y , we have $\widehat{\varepsilon}_n(y') = \widehat{\varepsilon}^m$ and $\widehat{\varepsilon}_{n'}(y') < \widehat{\varepsilon}_{n'}(y)$. The condition $U_{n'}(y) \geq U_n(y)$ can be expressed as

$$U_{n'}(y) - U_n(y) = \alpha y [\theta_{n'} H(\widehat{\varepsilon}_{n'}(y)) - \theta_n H(\widehat{\varepsilon}^m)] \geq 0,$$

which implies that $\theta_{n'} H(\widehat{\varepsilon}_{n'}(y)) - \theta_n H(\widehat{\varepsilon}^m) \geq 0$. Now consider the difference between $U_n(y')$ and $U_{n'}(y')$:

$$\begin{aligned} U_{n'}(y') - U_n(y') &= \alpha y' [\theta_{n'} H(\widehat{\varepsilon}_{n'}(y')) - \theta_n H(\widehat{\varepsilon}^m)] \\ &> \alpha y' [\theta_{n'} H(\widehat{\varepsilon}_{n'}(y)) - \theta_n H(\widehat{\varepsilon}^m)] \geq 0, \end{aligned}$$

where the first inequality uses the facts that $\widehat{\varepsilon}_{n'}(y') < \widehat{\varepsilon}_{n'}(y)$ and $H(\cdot)$ is decreasing, and the second inequality follows because $\theta_{n'} H(\widehat{\varepsilon}_{n'}(y)) - \theta_n H(\widehat{\varepsilon}^m) \geq 0$. This is a contradiction.

Case (3): $\widehat{\varepsilon}_{n'}(y) > \widehat{\varepsilon}_n(y) > \widehat{\varepsilon}^m$.

Subcase (3a): $\widehat{\varepsilon}_n(y') = \widehat{\varepsilon}_{n'}(y') = \widehat{\varepsilon}^m$. By an argument similar to that in case (1), we have $U_{n'}(y') - U_n(y') > 0$. This is a contradiction.

Subcase (3b): $\widehat{\varepsilon}^m < \widehat{\varepsilon}_n(y') < \widehat{\varepsilon}_n(y)$ and $\widehat{\varepsilon}^m < \widehat{\varepsilon}_{n'}(y') < \widehat{\varepsilon}_{n'}(y)$. It is enough to show that $\frac{\partial(U_{n'}(y) - U_n(y))}{\partial y} > 0$. More explicitly,

$$\frac{\partial(U_{n'}(y) - U_n(y))}{\partial y} = \alpha [\theta_{n'} H(\widehat{\varepsilon}_{n'}(y)) - \theta_n H(\widehat{\varepsilon}_n(y))] + \alpha [\theta_{n'} y \frac{\partial H(\widehat{\varepsilon}_{n'})}{\partial \widehat{\varepsilon}_{n'}} \frac{\partial \widehat{\varepsilon}_{n'}}{\partial y} - \theta_n y \frac{\partial H(\widehat{\varepsilon}_n)}{\partial \widehat{\varepsilon}_n} \frac{\partial \widehat{\varepsilon}_n}{\partial y}].$$

In the above equation, the first term is positive, as $U_{n'}(y) - U_n(y) \geq 0$. Thus, it is sufficient that the second term

$$\theta_{n'} y \frac{\partial H(\widehat{\varepsilon}_{n'})}{\partial \widehat{\varepsilon}_{n'}} \frac{\partial \widehat{\varepsilon}_{n'}}{\partial y} - \theta_n y \frac{\partial H(\widehat{\varepsilon}_n)}{\partial \widehat{\varepsilon}_n} \frac{\partial \widehat{\varepsilon}_n}{\partial y} > 0.$$

By $\frac{\partial H(\widehat{\varepsilon})}{\partial \widehat{\varepsilon}} = \frac{g(\widehat{\varepsilon})}{[r + \alpha(1 - G(\widehat{\varepsilon}))]^2} \Gamma(\widehat{\varepsilon})$ and $\frac{\partial \widehat{\varepsilon}_n}{\partial y} = -\frac{V_n}{y^2 \theta_n} = -\frac{1}{y\gamma} \widehat{\varepsilon}_n(y)$, we have

$$\begin{aligned} &\theta_{n'} y \frac{\partial H(\widehat{\varepsilon}_{n'})}{\partial \widehat{\varepsilon}_{n'}} \frac{\partial \widehat{\varepsilon}_{n'}}{\partial y} - \theta_n y \frac{\partial H(\widehat{\varepsilon}_n)}{\partial \widehat{\varepsilon}_n} \frac{\partial \widehat{\varepsilon}_n}{\partial y} \\ &\propto \theta_n \frac{g(\widehat{\varepsilon}_n) \widehat{\varepsilon}_n}{[r + \alpha(1 - G(\widehat{\varepsilon}_n))]^2} \Gamma(\widehat{\varepsilon}_n) - \theta_{n'} \frac{g(\widehat{\varepsilon}_{n'}) \widehat{\varepsilon}_{n'}}{[r + \alpha(1 - G(\widehat{\varepsilon}_{n'}))]^2} \Gamma(\widehat{\varepsilon}_{n'}) \end{aligned}$$

By previous results, $\Gamma(\widehat{\varepsilon}_{n'}) < \Gamma(\widehat{\varepsilon}_n) < 0$. Therefore, it is enough to show that

$$\theta_{n'} \frac{g(\widehat{\varepsilon}_{n'}) \widehat{\varepsilon}_{n'}}{[r + \alpha(1 - G(\widehat{\varepsilon}_{n'}))]^2} - \theta_n \frac{g(\widehat{\varepsilon}_n) \widehat{\varepsilon}_n}{[r + \alpha(1 - G(\widehat{\varepsilon}_n))]^2} \geq 0. \quad (12)$$

The following condition is sufficient for inequality (12): $g(\varepsilon)\varepsilon$ is weakly increasing in ε for all ε . More explicitly, $\frac{g'(\varepsilon)}{g(\varepsilon)} + \frac{1}{\varepsilon} \geq 0$. The logconcavity of $g(\varepsilon)$ means that $\frac{g'(\varepsilon)}{g(\varepsilon)}$ is decreasing in ε . Thus the LHS of the above inequality is decreasing in ε , and condition (4) is sufficient.

Subcase (3c): $\widehat{\varepsilon}_n(y') = \widehat{\varepsilon}^m$, and $\widehat{\varepsilon}^m < \widehat{\varepsilon}_{n'}(y') < \widehat{\varepsilon}_{n'}(y)$.

$$U_{n'}(y) - U_n(y) = \alpha y [\theta_{n'} H(\widehat{\varepsilon}_{n'}(y)) - \theta_n H(\widehat{\varepsilon}_n(y))] \geq 0.$$

$$U_{n'}(y') - U_n(y') = \alpha y' [\theta_{n'} H(\widehat{\varepsilon}_{n'}(y')) - \theta_n H(\widehat{\varepsilon}^m)].$$

By the continuity of $\widehat{\varepsilon}_n(y)$ in y , we can find a type $y'' \in (y, y']$ such that $\frac{V_n}{\theta_n y''} = \widehat{\varepsilon}^m$. Then between types y and y'' , we can apply the result of subcase (2) and get $U_{n'}(y'') - U_n(y'') > 0$. Between types y'' and y' , we can apply subcase (3a) and get $U_{n'}(y') - U_n(y') > U_{n'}(y'') - U_n(y'') > 0$. ■

Proof of Lemma 4.

Proof. Part (i). Consider the scenario in which \widehat{y}_n decreases to $\widehat{y}_n^L < \widehat{y}_n$ while \widehat{y}_{n-1} remains the same. Suppose after the change type n women adopt $\widehat{\varepsilon}_n^L(y)$ as the acceptance cutoff, which yields the payoff V_n^L . Notice that if type n women choose the threshold optimally, then V_n^L is the equilibrium payoff. Otherwise, V_n^L is weakly lower than the equilibrium payoff. By (3), we have

$$rX_n V_n^L = \alpha \int_{\widehat{y}_n^L}^{\widehat{y}_{n-1}} \int_{\widehat{\varepsilon}_n^L(y)}^{1+\gamma} (\theta_n y \varepsilon - V_n^L) dG(\varepsilon) dF(y).$$

Taking the difference between V_n^L and V_n and rearranging, we obtain

$$\begin{aligned} rX_n (V_n^L - V_n) + \alpha \int_{\widehat{y}_n^L}^{\widehat{y}_{n-1}} \int_{\widehat{\varepsilon}_n^L(y)}^{1+\gamma} (V_n^L - V_n) dG(\varepsilon) dF(y) \\ = \alpha \int_{\widehat{y}_n^L}^{\widehat{y}_n} \int_{\widehat{\varepsilon}_n(y)}^{1+\gamma} (\theta_n y \varepsilon - V_n) dG(\varepsilon) dF(y) - \alpha \int_{\widehat{y}_n^L}^{\widehat{y}_{n-1}} \int_{\widehat{\varepsilon}_n(y)}^{\widehat{\varepsilon}_n^L(y)} (\theta_n y \varepsilon - V_n) dG(\varepsilon) dF(y). \end{aligned}$$

If initially $\widehat{\varepsilon}_n(\widehat{y}_n) < 1 + \gamma$, or $\widehat{y}_n > \frac{V_n}{\theta_n(1+\gamma)}$, then consider the case in which a type n woman adopts the threshold such that $\widehat{\varepsilon}_n^L(y) = \widehat{\varepsilon}_n(y)$. The above equation implies $V_n^L > V_n$. Since type n women's equilibrium payoff is weakly higher than V_n^L , we conclude that a decrease in \widehat{y}_n leads to a strict increase in V_n .

If initially $\widehat{\varepsilon}_n(\widehat{y}_n) = 1 + \gamma$, then V_n must remain constant. Otherwise, the two sides of the above equation would have opposite signs.

Part (ii). Consider the scenario in which \widehat{y}_{n-1} increases to $\widehat{y}_{n-1}^H > \widehat{y}_{n-1}$ while \widehat{y}_n remains the same. Suppose type n women adopt the same acceptance threshold as before and obtain the payoff V_n^H . Since the threshold is not necessarily optimal given \widehat{y}_{n-1}^H , V_n^H is weakly lower than the equilibrium payoff. Similar to part (i), we can rearrange the difference between V_n^H and V_n as follows:

$$rX_n (V_n^H - V_n) + \alpha \int_{\widehat{y}_n}^{\widehat{y}_{n-1}^H} \int_{\widehat{\varepsilon}_n(y)}^{1+\gamma} (V_n^H - V_n) dG(\varepsilon) dF(y) = \alpha \int_{\widehat{y}_{n-1}}^{\widehat{y}_{n-1}^H} \int_{\widehat{\varepsilon}_n(y)}^{1+\gamma} (\theta_n y \varepsilon - V_n) dG(\varepsilon) dF(y).$$

The right-hand side of the above equation is strictly positive. Therefore, V_n strictly increases as the upper bound \hat{y}_{n-1} increases. ■

Proof of Lemma 5.

Proof. Part (i). Denote $\bar{S}(\hat{y}_1)$ as a segmentation with cutoff \hat{y}_1 , and $U_i(x; \bar{S})$ as type x men's expected payoff in submarket i given segmentation \bar{S} . We want to trace the $U_1(\hat{y}_1; \bar{S}(\hat{y}_1))$ curve and $U_2(\hat{y}_1; \bar{S}(\hat{y}_1))$ curve as \hat{y}_1 varies between \hat{y}_2 and \bar{y} . By Lemma 4, V_1 is weakly decreasing in \hat{y}_1 , and thus $\hat{\varepsilon}_1(\hat{y}_1; \bar{S}(\hat{y}_1))$ is decreasing in \hat{y}_1 . Therefore, $U_1(\hat{y}_1; \bar{S}(\hat{y}_1))$ is increasing in \hat{y}_1 . Again by Lemma 4, V_2 is increasing in \hat{y}_1 . Define $\Delta U(\hat{y}_1) \equiv U_1(\hat{y}_1; \bar{S}(\hat{y}_1)) - U_2(\hat{y}_1; \bar{S}(\hat{y}_1))$. It is easy to see that $\Delta U(\hat{y}_1)$ is continuous in \hat{y}_1 . In addition, $\Delta U(\bar{y}) > 0$, since $\bar{U}_1(\bar{y}) > U_2(\bar{y}; \bar{S}(\bar{y}))$ (when all men are participating in submarket 2), which follows $\bar{U}_1(\cdot) > \bar{U}_2(\cdot)$. So we only need to consider the following two cases.

Case 1. $\Delta U(\hat{y}_1)$ is always strictly positive for any $\hat{y}_1 \in (\hat{y}_2, \bar{y})$. Then a unique partial equilibrium exists with $y_1^* = \hat{y}_2$ (a corner equilibrium with all men in the range of $[\hat{y}_2, \bar{y}]$ participating in submarket 1).

Case 2. $\Delta U(\hat{y}_1)$ is strictly negative for some $\hat{y}_1 \in (\hat{y}_2, \bar{y})$. Then by continuity, there must exist some $\hat{y}_1 \in (\hat{y}_2, \bar{y})$ such that $\Delta U(\hat{y}_1) = 0$, and \hat{y}_1 is the (interior) equilibrium cutoff. This proves the existence of an interior partial equilibrium.

Now we show the uniqueness of equilibrium in case 2. Suppose there are two interior equilibria: y_1^* and $y_1^{*'}$, with $y_1^{*'} > y_1^*$. By Lemma 4, we have $U_1(y_1^{*'}; \bar{S}(y_1^{*'})) > U_1(y_1^*; \bar{S}(y_1^*))$ ($y_1^{*'}$ being an equilibrium marginal type means $\hat{\varepsilon}_1(y_1^{*'}; \bar{S}(y_1^{*'})) > \hat{\varepsilon}^m$). Again by Lemma 4, $U_2(y_1^{*'}; \bar{S}(y_1^{*'})) \leq U_2(y_1^*; \bar{S}(y_1^*))$ ($V_2' > V_2$, and hence $\hat{\varepsilon}_2(y_1^{*'}; \bar{S}(y_1^{*'})) \geq \hat{\varepsilon}_2(y_1^*; \bar{S}(y_1^*))$). Combining with the equilibrium indifference condition $U_2(y_1^{*'}; \bar{S}(y_1^{*'})) = U_1(y_1^{*'}; \bar{S}(y_1^{*'}))$, we have $U_2(y_1^{*'}; \bar{S}(y_1^{*'})) > U_1(y_1^*; \bar{S}(y_1^*))$. But $U_2(y_1^*; \bar{S}(y_1^*)) = U_1(y_1^*; \bar{S}(y_1^*))$ by the equilibrium indifference condition, which implies $U_2(y_1^*; \bar{S}(y_1^*)) < U_1(y_1^{*'}; \bar{S}(y_1^{*'}))$ (the single crossing property and $y_1^{*' > y_1^*$). This is a contradiction. Therefore, the interior partial equilibrium must be unique.

Part (ii). Consider $\hat{y}_2' > \hat{y}_2$. We use the superscript $'$ to indicate endogenous variables with \hat{y}_2' . Suppose $y_1^{*'} < y_1^*$. Then by Lemma 4, we have $U_1(y_1^*; \bar{S}(y_1^*)) \geq U_1'(y_1^{*'}; \bar{S}(y_1^{*'}))$ and $U_2(y_1^*; \bar{S}(y_1^*)) \leq U_2'(y_1^{*'}; \bar{S}(y_1^{*'}))$. Combining with the equilibrium condition $U_1(y_1^*; \bar{S}(y_1^*)) = U_2(y_1^*; \bar{S}(y_1^*))$, it must be the case that $U_2'(y_1^{*'}; \bar{S}(y_1^{*'})) \geq U_1'(y_1^{*'}; \bar{S}(y_1^{*'}))$. On the other hand, the equilibrium condition $U_1'(y_1^{*'}; \bar{S}(y_1^{*'})) = U_2'(y_1^{*'}; \bar{S}(y_1^{*'}))$ together with the presumption $y_1^{*'} < y_1^*$, due to the single crossing property, imply that $U_2'(y_1^{*'}; \bar{S}(y_1^{*'})) < U_1'(y_1^{*'}; \bar{S}(y_1^{*'}))$. This is a contradiction.

Part (iii). We know from part (ii) that y_1^* weakly decreases as \hat{y}_2 decreases. If y_1^* remains unchanged, then V_2^* weakly increases as indicated by Lemma 4. Next consider the case that y_1^* strictly decreases as \hat{y}_2 decreases. In this case V_1^* must strictly increase. This is because $\hat{\varepsilon}_1(y_1^*)$ must be strictly less than $1 + \gamma$. If $\hat{\varepsilon}_1(y_1^*) = 1 + \gamma$, then type y_1^* could have participated in

submarket 2, where he will be accepted with a positive probability as he would be the highest type man there. Now suppose V_2^* weakly decreases. Since V_1^* strictly increases, type y_1^* men, who were indifferent between submarket 1 and 2 in the original equilibrium, now strictly prefer submarket 2. By the single crossing property, this means that y_1^* should strictly increase. This contradicts the fact that y_1^* strictly decreases. ■

Proof of Lemma 6.

Proof. Part (i). Fix a \hat{y}_{n+1} . Denote $\bar{S}(\hat{y}_n)$ as a segmentation with threshold \hat{y}_n . Note that a given \hat{y}_n , by the presumption, will induce a unique partial equilibrium segmentation in submarkets 1, ..., n . And this has been incorporated in $\bar{S}(\hat{y}_n)$. Again, we focus on the indifference condition between submarkets n and $n+1$. We want to trace the $U_n(\hat{y}_n; \bar{S}(\hat{y}_n))$ curve and the $U_{n+1}(\hat{y}_n; \bar{S}(\hat{y}_n))$ curve as \hat{y}_n varies. Note that $U_n(\hat{y}_n; \bar{S}(\hat{y}_n))$ depends on the partial equilibrium segmentation in submarkets 1, ..., n given \hat{y}_n . Again, we consider two cases.

Case 1. Suppose $U_n(\hat{y}_{n+1}; \bar{S}(\hat{y}_{n+1})) \geq \bar{U}_{n+1}(\hat{y}_{n+1})$. Then a unique partial equilibrium exists, with no men participating in submarket $n+1$ (a corner equilibrium).

Case 2. Suppose $U_n(\hat{y}_{n+1}; \bar{S}(\hat{y}_{n+1})) < \bar{U}_{n+1}(\hat{y}_{n+1})$. But by Lemma 4, we have $\bar{U}_n(\bar{y}) > U_{n+1}(\bar{y}; \bar{S}(\bar{y}))$. By continuity, the $U_{n+1}(\hat{y}_n; \bar{S}(\hat{y}_n))$ curve and the $U_n(\hat{y}_n; \bar{S}(\hat{y}_n))$ curve must have at least one intersection within the domain (\hat{y}_{n+1}, \bar{y}) . Given any intersection point, \hat{y}_i^* ($i < n$) exists and is unique according to the presumption of the previous steps. This establishes the existence of an interior partial equilibrium.

Now we establish the uniqueness of equilibrium in case 2. Suppose there are two interior equilibria: y_n^* and y_n^* , with $y_n^* > y_n^*$. By property (iii) for n , we have $V_n^* \geq V_n^*$. Therefore, $U_n(y_n^*; \bar{S}(y_n^*)) \geq U_n(y_n^*; \bar{S}(y_n^*))$. In addition, by Lemma 4 $y_n^* > y_n^*$ implies $V_{n+1}^* > V_{n+1}^*$. As a result, $U_{n+1}(y_n^*; \bar{S}(y_n^*)) \leq U_{n+1}(y_n^*; \bar{S}(y_n^*))$. The indifference condition tells us that $U_n(y_n^*; \bar{S}(y_n^*)) = U_{n+1}(y_n^*; \bar{S}(y_n^*))$. Combined with the above two inequalities, we obtain $U_{n+1}(y_n^*; \bar{S}(y_n^*)) \geq U_n(y_n^*; \bar{S}(y_n^*))$. On the other hand, the other indifference condition $U_n(y_n^*; \bar{S}(y_n^*)) = U_{n+1}(y_n^*; \bar{S}(y_n^*))$ together with the presumption that $y_n^* > y_n^*$, due to the single crossing property, imply $U_{n+1}(y_n^*; \bar{S}(y_n^*)) < U_n(y_n^*; \bar{S}(y_n^*))$. This is a contradiction.

Part (ii). Consider $\hat{y}'_{n+1} > \hat{y}_{n+1}$. Suppose $y_n^* < y_n^*$. Then by Lemma 4, $V_{n+1}^* < V_{n+1}^*$ and hence $U_{n+1}(y_n^*; \bar{S}(y_n^*)) \leq U_{n+1}(y_n^*; \bar{S}(y_n^*))$. By property (iii) for n , $y_n^* < y_n^*$ implies $V_n^* > V_n^*$. Therefore, $U_n(y_n^*; \bar{S}(y_n^*)) \geq U_n(y_n^*; \bar{S}(y_n^*))$. By the indifference condition $U_{n+1}(y_n^*; \bar{S}(y_n^*)) = U_n(y_n^*; \bar{S}(y_n^*))$, the above two inequalities suggest $U_{n+1}(y_n^*; \bar{S}(y_n^*)) \geq U_n(y_n^*; \bar{S}(y_n^*))$. On the other hand, the equilibrium condition $U'_{n+1}(y_n^*; \bar{S}(y_n^*)) = U'_n(y_n^*; \bar{S}(y_n^*))$ together with the presumption $y_n^* < y_n^*$, due to the single crossing property, imply that $U'_{n+1}(y_n^*; \bar{S}(y_n^*)) < U'_n(y_n^*; \bar{S}(y_n^*))$. This is a contradiction.

Part (iii). First, if y_n^* remains the same as \hat{y}_{n+1} decreases, then by Lemma 4 V_{n+1}^* weakly increases. Second, if y_n^* strictly decreases as \hat{y}_{n+1} decreases, then V_n^* strictly increases by

property (iii) for n . The rest of the proof is similar to that of part (iii) of Lemma 5, and thus is omitted. ■

Proof of Lemma 7.

Proof. Part (i). By the value function of V_n , equation (3), we have

$$V_n = \int_{y_n} \frac{\alpha \theta_n y \int_{\widehat{\varepsilon}_n(y)}^{1+\gamma} \varepsilon dG(\varepsilon)}{r/q_n + \alpha \int_{y_n} (1 - G(\widehat{\varepsilon}_n(y))) dF_n(y)} dF_n(y) \equiv \int_{y_n} Z_n(y) dF_n(y)$$

Suppose α increases to α' . By previous results, men's acceptance cutoff increases to $\widehat{\varepsilon}^{m'} > \widehat{\varepsilon}^m$. Assume that, under α' , women in submarket n adopt the same acceptance cutoff as under α . Denote $\widehat{\varepsilon}_n^a(y)$ as the resulting threshold for any $y \in \{y_n\}$, $Z_n^a(y)$ as the integrand under α' and $\widehat{\varepsilon}_n^a(y)$, and $V_n(\alpha')$ as the value of type n women under $Z_n^a(y)$ and α' . Note that the resulting $V_n(\alpha')$ is not optimal for women under α' .

First, consider the case that $\widehat{\varepsilon}_n(y) > \widehat{\varepsilon}^m$ (the lower types). Then it is straightforward that $Z_n^a(y) \geq Z_n(y)$, as

$$\begin{aligned} Z_n^a(y) - Z_n(y) &\propto \frac{\alpha' \int_{\widehat{\varepsilon}_n(y)} \varepsilon dG(\varepsilon)}{r/q_n + \alpha' \int_{y_n} [1 - G(\widehat{\varepsilon}_n^a(y))] dF_n(y)} - \frac{\alpha \int_{\widehat{\varepsilon}_n(y)} \varepsilon dG(\varepsilon)}{r/q_n + \alpha \int_{y_n} [1 - G(\widehat{\varepsilon}_n(y))] dF_n(y)} \\ &> \frac{\alpha' \int_{\widehat{\varepsilon}_n(y)} \varepsilon dG(\varepsilon)}{r/q_n + \alpha' \int_{y_n} [1 - G(\widehat{\varepsilon}_n(y))] dF_n(y)} - \frac{\alpha \int_{\widehat{\varepsilon}_n(y)} \varepsilon dG(\varepsilon)}{r/q_n + \alpha \int_{y_n} [1 - G(\widehat{\varepsilon}_n(y))] dF_n(y)}, \end{aligned}$$

where the inequality follows because $\widehat{\varepsilon}^{m'} > \widehat{\varepsilon}^m$.

Next, consider y with $\widehat{\varepsilon}_n(y) = \widehat{\varepsilon}^m$ (the higher types). Also denote the probability that, among all $\tilde{y} \in y_n$, $\widehat{\varepsilon}_n(\tilde{y}) = \widehat{\varepsilon}^m$ as \mathbb{P} . Then

$$\begin{aligned} &\lim_{\alpha' \rightarrow \alpha} \frac{Z_n^a(y) - Z_n(y)}{\alpha' - \alpha} \\ &\propto \frac{r}{q_n} \int_{\widehat{\varepsilon}^m}^{1+\gamma} \varepsilon dG(\varepsilon) + \alpha g(\widehat{\varepsilon}^m) \left[-\frac{r}{q_n} \widehat{\varepsilon}^m - \alpha \widehat{\varepsilon}^m \int_{y_n} [1 - G(\widehat{\varepsilon}_n(y))] dF_n(y) + \alpha \int_{\widehat{\varepsilon}^m} \varepsilon dG(\varepsilon) \mathbb{P} \right] \frac{\partial \widehat{\varepsilon}^m}{\partial \alpha} \\ &> \frac{r}{q_n} \int_{\widehat{\varepsilon}^m}^{1+\gamma} \varepsilon dG(\varepsilon) + \alpha g(\widehat{\varepsilon}^m) \left[-\frac{r}{q_n} \widehat{\varepsilon}^m - \alpha \widehat{\varepsilon}^m [1 - G(\widehat{\varepsilon}^m)] + \alpha \int_{\widehat{\varepsilon}^m}^{1+\gamma} \varepsilon dG(\varepsilon) \mathbb{P} \right] \frac{\partial \widehat{\varepsilon}^m}{\partial \alpha} \\ &= \frac{r}{q_n} \int_{\widehat{\varepsilon}^m}^{1+\gamma} \varepsilon dG(\varepsilon) + \alpha g(\widehat{\varepsilon}^m) \left[r \left(1 - \frac{1}{q_n}\right) \widehat{\varepsilon}^m - (1 - \mathbb{P}) \alpha \int_{\widehat{\varepsilon}^m} \varepsilon dG(\varepsilon) \right] \frac{\partial \widehat{\varepsilon}^m}{\partial \alpha} \end{aligned}$$

Let $q_n \geq 1$. After plugging in $\frac{\partial \widehat{\varepsilon}^m}{\partial \alpha}$, a sufficient condition for $Z_n^a(y) - Z_n(y) \geq 0$ is

$$\frac{r}{\alpha} \frac{r + \alpha [1 - G(\widehat{\varepsilon}^m)]}{g(\widehat{\varepsilon}^m)} - \alpha \int_{\widehat{\varepsilon}^m} (\varepsilon - \widehat{\varepsilon}^m) dG(\varepsilon) \geq 0. \quad (13)$$

Therefore, if (13) holds and q_n is larger than 1 or close to 1 (by continuity), then $Z_n^a(y) - Z_n(y) \geq 0$. Combined with the previous discussion, this further implies that $V_n' > V_n(\alpha') > V_n$.

The remaining task is to find sufficient conditions for (13) to hold. The second term in (13), $\alpha \int_{\widehat{\varepsilon}^m} (\varepsilon - \widehat{\varepsilon}^m) dG(\varepsilon)$, is decreasing in $\widehat{\varepsilon}^m$. Thus it is less than $\alpha\gamma$ (when $\widehat{\varepsilon}^m = 1 - \gamma$). Also, by Assumption 3 the first term in (13) is decreasing in $\widehat{\varepsilon}^m$. Therefore, with the assumption that $\widehat{\varepsilon}^m \leq 1$, the following condition is sufficient for (13)

$$\frac{r}{\alpha} \frac{r + \alpha[1 - G(1)]}{g(1)} - \alpha\gamma \geq 0.$$

Part (ii). For the high types in submarket n such that $\widehat{\varepsilon}_n(y) = \widehat{\varepsilon}^m$, by the Envelop Theorem it can readily be computed that

$$\frac{\partial U_n(y)}{\partial \alpha} = \frac{\frac{r}{\alpha} U_n(y_n)}{r + \alpha[1 - G(\widehat{\varepsilon}^m)]} > 0,$$

which is obviously increasing in y . Therefore, $\Delta U_n(y)$ increases in y whenever $\widehat{\varepsilon}_n(y) = \widehat{\varepsilon}^m$.

Part (iii). First, we show that $\Delta U_n(y_n^*) > 0$. Following the expression of $\frac{\partial U_n(y)}{\partial \alpha}$ in part (ii), this is obvious as $\widehat{\varepsilon}_n(y_n^*) = \widehat{\varepsilon}^m$.

Next we show $\Delta U_n(y_n^*) < \Delta U_{n+1}(y_n^*)$. By previous results, we have

$$\frac{\partial U_n(y_n^*)}{\partial \alpha} = \frac{\frac{r}{\alpha} U_n(y_n^*) - \alpha[V_n - U_n(y)]g(\widehat{\varepsilon}_n) \frac{\widehat{\varepsilon}_n}{V_n} \frac{\partial V_n}{\partial \alpha}}{r + \alpha[1 - G(\widehat{\varepsilon}_n)]},$$

where $\widehat{\varepsilon}_n$ denotes $\widehat{\varepsilon}_n = \widehat{\varepsilon}_n(y_n^*) < \widehat{\varepsilon}^m$, and

$$\frac{\partial U_{n+1}(y_n^*)}{\partial \alpha} = \frac{\frac{r}{\alpha} U_{n+1}(y_n^*)}{r + \alpha[1 - G(\widehat{\varepsilon}^m)]} = \frac{\frac{r}{\alpha} U_n(y_n^*)}{r + \alpha[1 - G(\widehat{\varepsilon}^m)]} > 0.$$

If $\frac{\partial U_n(y_n^*)}{\partial \alpha} \leq 0$, then the claim is trivially satisfied. Now suppose $\frac{\partial U_n(y_n^*)}{\partial \alpha} \geq 0$. Since $\frac{\partial V_n}{\partial \alpha} > \frac{\partial U_n(y_n^*)}{\partial \alpha}$,

$$\frac{\partial U_n(y_n^*)}{\partial \alpha} < \frac{\frac{r}{\alpha} U_n(y_n^*) - \alpha[V_n - U_n(y)]g(\widehat{\varepsilon}_n) \frac{\widehat{\varepsilon}_n}{V_n} \frac{\partial U_n(y_n^*)}{\partial \alpha}}{r + \alpha[1 - G(\widehat{\varepsilon}_n)]},$$

which means

$$\frac{\partial U_n(y_n^*)}{\partial \alpha} < \frac{\frac{r}{\alpha} U_n(y_n^*)}{r + \alpha[1 - G(\widehat{\varepsilon}_n)] + \alpha[V_n - U_n(y_n^*)]g(\widehat{\varepsilon}_n) \frac{\widehat{\varepsilon}_n}{V_n}}.$$

Now, we have

$$\frac{\partial U_n(y_n^*)}{\partial \alpha} - \frac{\partial U_{n+1}(y_n^*)}{\partial \alpha} < 0 \Leftrightarrow [G(\widehat{\varepsilon}_n) - G(\widehat{\varepsilon}^m)] - [1 - \frac{U_n(y_n^*)}{V_n}]g(\widehat{\varepsilon}_n)\widehat{\varepsilon}_n \leq 0, \quad (14)$$

which is equivalent to

$$1 - \frac{U_n(y_n^*)}{V_n} - \frac{G(\widehat{\varepsilon}_n) - G(\widehat{\varepsilon}^m)}{g(\widehat{\varepsilon}_n)\widehat{\varepsilon}_n} \geq 0.$$

Define

$$B_n(y) \equiv 1 - \frac{U_n(y)}{V_n} - \frac{G(\widehat{\varepsilon}_n(y)) - G(\widehat{\varepsilon}^m)}{g(\widehat{\varepsilon}_n(y))\widehat{\varepsilon}_n(y)}.$$

Then (14) requires $B_n(y_n^*) \geq 0$. Observing $B_n(y)$, we first notice that $B_n(y^A) = 0$, where y^A is the type of man such that $U_n(y^A) = V_n$. Therefore, a sufficient condition for the above inequality to hold is that $B_n'(y) < 0$ whenever $B_n(y) < 0$. Using the fact that $1 - \frac{U_n}{V_n} < \frac{G(\hat{\varepsilon}_n) - G(\hat{\varepsilon}^m)}{g(\hat{\varepsilon}_n)\hat{\varepsilon}_n}$ when $B_n(y) < 0$, we get

$$\begin{aligned} B_n'(y) &\propto -\left[1 - \frac{U_n}{V_n}\right] \left[\frac{\alpha g(\hat{\varepsilon}_n)\hat{\varepsilon}_n}{r + \alpha[1 - G(\hat{\varepsilon}_n)]} - 1\right] - \frac{G(\hat{\varepsilon}_n) - G(\hat{\varepsilon}^m)}{g(\hat{\varepsilon}_n)\hat{\varepsilon}_n} \left[\frac{g'(\hat{\varepsilon}_n)\hat{\varepsilon}_n}{g(\hat{\varepsilon}_n)} + 1\right] \\ &< -\left[1 - \frac{U_n}{V_n}\right] \left[\frac{\alpha g(\hat{\varepsilon}_n)\hat{\varepsilon}_n}{r + \alpha[1 - G(\hat{\varepsilon}_n)]} + \frac{g'(\hat{\varepsilon}_n)\hat{\varepsilon}_n}{g(\hat{\varepsilon}_n)}\right] \\ &\propto -\frac{\alpha g^2(\hat{\varepsilon}_n)}{r + \alpha[1 - G(\hat{\varepsilon}_n)]} - g'(\hat{\varepsilon}_n) \propto \left[-\frac{g(\hat{\varepsilon}_n)}{r + \alpha[1 - G(\hat{\varepsilon}_n)]}\right]', \end{aligned}$$

which is negative by Assumption 3. ■

Discussion on the conditions of Lemma 7

The following lemma provides explicit conditions under which the conditions for part (i) of Lemma 7 hold.

Lemma 9 *If ε follows either a truncated normal or uniform distribution, then there exists an R , such that when $\frac{r}{\alpha} \geq R$, $\hat{\varepsilon}^m \leq 1$ and $\frac{r + \alpha[1 - G(1)]}{g(1)} - \frac{\alpha^2 \gamma}{r} \geq 0$ both hold.*

Proof. We have already shown that $\hat{\varepsilon}^m$ increases in α . It is easy to see that it also decreases in r . Therefore, $\hat{\varepsilon}^m$ is decreasing in r/α . Moreover, the condition $\frac{r + \alpha[1 - G(1)]}{g(1)} - \frac{\alpha^2 \gamma}{r} \geq 0$ can be simplified as

$$\begin{aligned} \frac{r}{\alpha} \left(\frac{r}{\alpha} + \frac{1}{2}\right) \frac{\sigma}{\phi(0)} \left[\Phi\left(\frac{\gamma}{\sigma}\right) - \Phi\left(\frac{-\gamma}{\sigma}\right)\right] &\geq \gamma, \text{ when } \varepsilon \text{ follows truncated normal distribution,} \\ \frac{r}{\alpha} \left(\frac{r}{\alpha} + \frac{1}{2}\right) &\geq \frac{1}{2}, \text{ when } \varepsilon \text{ follows uniform distribution.} \end{aligned}$$

In either case, this condition is satisfied if $\frac{r}{\alpha}$ is not too small.

Actually, under uniform distribution, the conditions can be weakened. Note that a sufficient for condition (13) is that

$$\frac{r}{\alpha} \frac{r + \alpha[1 - G(\tilde{\varepsilon})]}{g(\tilde{\varepsilon})} - \alpha \int_{\tilde{\varepsilon}} (\varepsilon - \tilde{\varepsilon}) dG(\varepsilon) \geq 0 \text{ for any } \tilde{\varepsilon} \in [1 - \gamma, 1 + \gamma]. \quad (15)$$

Under uniform distribution, the derivative of the left-hand side of (15) equals to $-r + \alpha[1 - G(\tilde{\varepsilon})]$, and the second derivative is $-\frac{\alpha}{2\gamma} < 0$. This means that the left-hand side is either always decreasing in $\tilde{\varepsilon}$ or first increasing then decreasing in $\tilde{\varepsilon}$. Therefore, as long as the inequality holds for the two end points, (15) is satisfied. When $\tilde{\varepsilon} = 1 + \gamma$, the inequality is trivially satisfied. When $\tilde{\varepsilon} = 1 - \gamma$, the inequality is equivalent to $\frac{r}{\alpha} \left(\frac{r}{\alpha} + 1\right) \geq 1/2$. Therefore, when ε is uniformly distributed, a sufficient condition is $\frac{r}{\alpha} \left(\frac{r}{\alpha} + 1\right) \geq 1/2$. ■

The condition for part (iii) of Lemma 7 to hold is $\frac{\partial V_n}{\partial \alpha} > \frac{\partial U_n(y_n^*)}{\partial \alpha}$. This condition is not restrictive, and we expect it to hold fairly generally when $q_n \geq 1$ or q_n is close enough to 1. Our reasoning is as follows. We already know that when $q_n = 1$, V_n can be expressed as a weighted average of $U_n(y)$: $V_n = \int_{y_n} w_n(y) U_n(y) dF_n(y)$. The result that $\frac{\partial V_n}{\partial \alpha} > \frac{\partial U_n(y_n^*)}{\partial \alpha}$ is due to two things. First, higher type men will generally benefit more from an increase in α : $\frac{\partial U_n(y)}{\partial \alpha}$ is increasing in y . This is because their expected payoffs per meeting are higher, which is due to their higher vertical types and higher acceptance probabilities. Moreover, their increase in the acceptance cutoffs induced by an increase in V_n are smaller: $\Delta \widehat{\varepsilon}_n^w(y) = \frac{V'_n - V_n}{\theta_n y}$ is decreasing in y .⁶¹ Second, as α increases, the quasi-weights $w_n(y)$ increase for higher types but decrease for lower types, which tends to make V_n increase faster than $U_n(y_n^*)$ does. The following Lemma provides a set of conditions that are sufficient (far from being necessary) for $\frac{\partial V_n}{\partial \alpha} > \frac{\partial U_n(y_n^*)}{\partial \alpha}$.

Lemma 10 *Suppose ε is uniformly distributed on $[1 - \gamma, 1 + \gamma]$, q_n is close to 1, and $\frac{1+\gamma}{\gamma} - 4\left(\frac{r}{\alpha}\right)^2 \frac{\gamma}{1+\gamma} - 4\frac{r}{\alpha} \leq 0$.⁶² Then, $\frac{\partial V_n}{\partial \alpha} > \frac{\partial U_n(y_n^*)}{\partial \alpha}$.*

Proof. We first show that, for $y \in y_n$ and $\widehat{\varepsilon}_n^w(y) \geq \widehat{\varepsilon}_n^m$, $\frac{\partial U_n(y)}{\partial \alpha}$ is increasing in y . In particular, we have

$$\frac{\partial U_n(y)}{\partial \alpha} = \frac{\frac{r}{\alpha} U_n(y) - \alpha [V_n - U_n(y)] g(\widehat{\varepsilon}_n) \frac{\widehat{\varepsilon}_n}{V_n} \frac{\partial V_n}{\partial \alpha}}{r + \alpha [1 - G(\widehat{\varepsilon}_n)]},$$

where $\widehat{\varepsilon}_n$ is a shorthand for $\widehat{\varepsilon}_n(y)$. Given that ε is uniform and $\frac{\partial V_n}{\partial \alpha} > 0$, the second term is more negative for a smaller y . Thus, we only need to show that $\frac{U_n(y)}{r + \alpha [1 - G(\widehat{\varepsilon}_n)]}$ is increasing in y . This is equivalent to

$$\frac{\int_{\widehat{\varepsilon}_n}^{1+\gamma} \varepsilon dG(\varepsilon)}{g(\widehat{\varepsilon}_n) \widehat{\varepsilon}_n} + \widehat{\varepsilon}_n - 2 \frac{\alpha \int_{\widehat{\varepsilon}_n}^{1+\gamma} \varepsilon dG(\varepsilon)}{[r + \alpha (1 - G(\widehat{\varepsilon}_n))]} \geq 0. \quad (16)$$

Using the fact that $\widehat{\varepsilon}_n^m = \frac{\alpha \int_{\widehat{\varepsilon}_n^m}^{1+\gamma} \varepsilon dG(\varepsilon)}{[r + \alpha (1 - G(\widehat{\varepsilon}_n^m))]}$ and that $\frac{\alpha \int_{\widehat{\varepsilon}_n}^{1+\gamma} \varepsilon dG(\varepsilon)}{[r + \alpha (1 - G(\widehat{\varepsilon}_n))]}$ is decreasing in $\widehat{\varepsilon}_n$ for $\widehat{\varepsilon}_n \geq \widehat{\varepsilon}_n^m$, the following condition is sufficient for (16):

$$\frac{\frac{r}{\alpha} + (1 - G(\widehat{\varepsilon}_n))}{g(\widehat{\varepsilon}_n)} \geq \left(2 - \frac{\widehat{\varepsilon}_n}{\widehat{\varepsilon}_n^m}\right) \widehat{\varepsilon}_n.$$

Note that the LHS of the above inequality is decreasing in $\widehat{\varepsilon}_n$, and the RHS is less than $\widehat{\varepsilon}_n^m$. Thus the following condition is sufficient:

$$\frac{\frac{r}{\alpha} + (1 - G(\widehat{\varepsilon}_n^*))}{g(\widehat{\varepsilon}_n^*)} \geq \widehat{\varepsilon}_n^m,$$

⁶¹Of course, this does not translate directly to an increase in $\Delta U_n(y)$ for higher types, as it also depends on the distribution of ε .

⁶²Again, this condition holds if r/α is not too small.

where $\widehat{\varepsilon}_n^*$ is the highest $\widehat{\varepsilon}_n$: $\widehat{\varepsilon}_n(y_n^*) \equiv \widehat{\varepsilon}_n^*$. Under uniform distribution, this condition becomes

$$\frac{r}{\alpha}2\gamma + [(1 + \gamma) - \widehat{\varepsilon}_n^*] \geq \widehat{\varepsilon}_n^m.$$

The above condition holds if $\widehat{\varepsilon}_n^m \leq \frac{r}{\alpha}2\gamma$, which under uniform distribution is equivalent to

$$\frac{1 + \gamma}{\gamma} - 4\left(\frac{r}{\alpha}\right)^2 \frac{\gamma}{1 + \gamma} - 4\frac{r}{\alpha} \leq 0.$$

Again, this condition holds if r/α is not too small.

Combining with part (ii) of Lemma 7, we reach the conclusion that for all $y \in y_n$, $\frac{\partial U_n(y)}{\partial \alpha}$ is increasing in y .

Now we set $q_n = 1$ and show $\frac{\partial V_n}{\partial \alpha} > \frac{\partial U_n(y_n^*)}{\partial \alpha}$. Recall that when $q_n = 1$, we have $V_n = \int_{y_n} w_n(y)U_n(y)dF_n(y)$, where the quasi-weight $w_n(y)$ is defined in equation (7). Taking derivative of $w_n(y)$ with respect to α , we have

$$\begin{aligned} \frac{\partial w_n(y)}{\partial \alpha} &\propto [r + \alpha[1 - G(\widehat{\varepsilon}_n(y))]] \left\{ \frac{1 - G(\widehat{\varepsilon}_n(y))}{r + \alpha[1 - G(\widehat{\varepsilon}_n(y))]} - \frac{\alpha g(\widehat{\varepsilon}_n(y)) \frac{\partial \widehat{\varepsilon}_n(y)}{\partial \alpha}}{r + \alpha[1 - G(\widehat{\varepsilon}_n(y))]} \right. \\ &\quad \left. - \frac{\int_{y_n} [1 - G(\widehat{\varepsilon}_n(\tilde{y})) - \alpha g(\widehat{\varepsilon}_n(\tilde{y})) \frac{\partial \widehat{\varepsilon}_n(\tilde{y})}{\partial \alpha}] dF_n(\tilde{y})}{r + \alpha \int_{y_n} [1 - G(\widehat{\varepsilon}_n(\tilde{y}))] dF_n(\tilde{y})} \right\}. \end{aligned}$$

In the above expression, the third term in the bracket is independent of y . First consider the higher types of y with $\widehat{\varepsilon}_n(y) = \widehat{\varepsilon}_n^m$. It is obvious that all the terms in the above expression are constant in y . Therefore, $\frac{\partial w_n(y)}{\partial \alpha}$ is constant in y . Next consider the lower types of y such that $\widehat{\varepsilon}_n(y) = \widehat{\varepsilon}_n^w(y)$. Note that $[r + \alpha[1 - G(\widehat{\varepsilon}_n(y))]]$ is increasing in y and the first term in the bracket is also increasing in y . The second term in the bracket, due to uniform distribution, can be explicitly written as

$$\frac{\alpha \frac{1}{\theta_n y} \frac{\partial V_n}{\partial \alpha}}{2r\gamma + \alpha[1 + \gamma - \widehat{\varepsilon}_n(y)]},$$

which is decreasing in y . Therefore, $\frac{\partial w_n(y)}{\partial \alpha}$ is increasing in y .

In addition, $\frac{\partial \widehat{\varepsilon}_n(y)}{\partial \alpha}$ is continuous in y . In particular, it is continuous at $y = y^A$ ($\widehat{\varepsilon}_n^w(y^A) = \widehat{\varepsilon}_n^m$). Therefore, $\frac{\partial w_n(y)}{\partial \alpha}$ is also continuous in y . Combined with the earlier discussion, this implies that $\frac{\partial w_n(y)}{\partial \alpha}$ is increasing in y for $y \leq y^A$ and constant in y when $y \geq y^A$. Moreover, since $\int_{y_n} w_n(y)dF_n(y) = 1$ for any α , thus we have $\int_{y_n} \frac{\partial w_n(y)}{\partial \alpha} dF_n(y) = 0$. As a result, $\frac{\partial w_n(y)}{\partial \alpha}$ must be negative for smaller y and then become positive for larger y .

Notice that $w_n(y)$ can also be considered as a quasi-p.d.f. of the weight distribution. Then the above discussion shows that the quasi-weight distribution with a larger α first order stochastic dominates the one with a smaller α .

Finally, we have

$$\begin{aligned}
\frac{\partial V_n}{\partial \alpha} &= \int_{y_n} \frac{\partial w_n(y)}{\partial \alpha} U_n(y) dF_n(y) + \int_{y_n} w_n(y) \frac{\partial U_n(y)}{\partial \alpha} dF_n(y) \\
&\geq \int_{y_n} w_n(y) \frac{\partial U_n(y)}{\partial \alpha} dF_n(y) > \int_{y_n} w_n(y) \frac{\partial U_n(y_n^*)}{\partial \alpha} dF_n(y) \\
&= \frac{\partial U_n(y_n^*)}{\partial \alpha}.
\end{aligned}$$

In the above derivation, the first inequality is due to $\int_{y_n} \frac{\partial w_n(y)}{\partial \alpha} U_n(y) dF_n(y) \geq 0$, which follows the facts that $U_n(y)$ strictly increases in y , $\frac{\partial w_n(y)}{\partial \alpha}$ is increasing in y , and $\int_{y_n} \frac{\partial w_n(y)}{\partial \alpha} dF_n(y) = 0$. The second inequality holds because $\frac{\partial U_n(y)}{\partial \alpha}$ strictly increases in y . ■

Proof of Proposition 3.

Proof. Part (i). We use notations without superscript to denote variables before the increase in α , and use superscripts "' and "a" to indicate variables under α' when the segmentation is endogenously adjusted and when the segmentation stays constant, respectively.

We first show that y_1^* must strictly increase. Suppose, to the contrary, $y_1^{*'} \leq y_1^*$. By Lemma 7, $V_1^a > V_1$ and $U_2^a(y_1^*) - U_2(y_1^*) > U_1^a(y_1^*) - U_1(y_1^*)$. Since $U_2(y_1^*) = U_1(y_1^*)$, we have $U_2^a(y_1^*) > U_1^a(y_1^*)$. Since $y_1^{*'} \leq y_1^*$, Lemma 4 implies that $V_1' \geq V_1^a$. Thus $U_1'(y_1^*) \leq U_1^a(y_1^*)$. On the other hand, if in the new segmentation type y_1^* men deviate to submarket 2, they will still be the highest types there, which implies $U_2'(y_1^*) = U_2^a(y_1^*)$. Taken together, we have $U_1'(y_1^*) < U_2'(y_1^*)$, which means that men of type y_1^* would strictly prefer to participate in submarket 2. This is a contradiction, and thus y_1^* must strictly increase in α .

Next we show that y_2^* must strictly increase. Suppose $y_2^{*'} \leq y_2^*$. Since $y_1^{*'} > y_1^*$ and $y_2^{*'} \leq y_2^*$, Lemma 4 implies that $V_2' \geq V_2^a$. Then by the same argument as in the previous step, $U_3'(y_2^*) > U_2'(y_2^*)$, i.e., men of type y_2^* strictly prefer to participate in submarket 3. This is a contradiction. This argument can be readily extended to all thresholds in the lower classes.

Part (ii). Using the implicit function theorem, we can compute the percentage change of $U_n(y_n^*)$

$$\begin{aligned}
\frac{\partial U_n(y_n^*)}{\partial \alpha} \frac{\alpha}{U_n(y_n^*)} &= \frac{r - \frac{\alpha^2}{U_n(y_n^*)} [\theta_n y_n^* \widehat{\varepsilon}_n(y_n^*) - U_n(y_n^*)] g(\widehat{\varepsilon}_n(y_n^*)) \frac{\partial \widehat{\varepsilon}_n(y_n^*)}{\partial \alpha}}{\alpha [1 - G(\widehat{\varepsilon}_n(y_n^*))] + r}, \\
\frac{\partial U_{n+1}(y_n^*)}{\partial \alpha} \frac{\alpha}{U_{n+1}(y_n^*)} &= \frac{r}{\alpha [1 - G(\widehat{\varepsilon}^m)] + r}.
\end{aligned}$$

From part (i), we know that y_n^* increases in α . Therefore, $\frac{\partial U_n(y_n^*)}{\partial \alpha} \frac{\alpha}{U_n(y_n^*)}$ must be smaller than $\frac{\partial U_{n+1}(y_n^*)}{\partial \alpha} \frac{\alpha}{U_{n+1}(y_n^*)}$. Since $\widehat{\varepsilon}_n(y_n^*) > \widehat{\varepsilon}^m$, it implies that $\frac{\partial \widehat{\varepsilon}_n(y_n^*)}{\partial \alpha}$ must be strictly positive. This further implies that V_n strictly increases in α .

Part (iii). According to the previous results, $U_1^a(\bar{y}) > U_1(\bar{y})$. Since $y_1^{*'} > y_1^*$, following Lemma 4 we have $V_1' < V_1^a$, and thus $U_1'(\bar{y}) = U_1^a(\bar{y}) > U_1(\bar{y})$. The continuity implies that the highest types of men (close to the highest type \bar{y}) must be better off.

Now consider the highest type men y_n^* in submarket $n > 1$. Given that an increase in α is small, for type y_n^* in submarket $n + 1$, men's acceptance cutoff is still binding. Then we have $U_{n+1}'(y_n^*) = U_{n+1}^a(y_n^*) > U_{n+1}(y_n^*)$. This means that type y_n^* men are strictly better off after an increase in α . Then, by continuity, men with types slightly higher than y_n^* should also be strictly better off. This means that the lowest types of men originally in submarket n , $1 \leq n < N$, are strictly better off as well. ■

Proof of Proposition 4.

Proof. We first show that if the initial equilibrium segmentation $\{y_n^*\}$ stays unchanged, then V_n strictly decreases for all n . By the value function (3), we have

$$rV_n = \alpha \frac{Y_n}{X_n} \int_{y_n} \int_{\widehat{\varepsilon}_n(y)}^{1+\gamma} (\theta_n \varepsilon y - V_n) dG(\varepsilon) dF_n(y).$$

Suppose V_n weakly increases for some n . Then for any y with $\widehat{\varepsilon}_n(y) = \widehat{\varepsilon}^m$, $\int_{\widehat{\varepsilon}_n(y)}^{1+\gamma} (\theta_n \varepsilon y - V_n) dG(\varepsilon)$ weakly decreases, as V_n weakly increases and $\widehat{\varepsilon}^m$ stays unchanged. For any y with $\widehat{\varepsilon}_n(y) = \widehat{\varepsilon}_n^w(y)$, $\int_{\widehat{\varepsilon}_n(y)}^{1+\gamma} (\theta_n \varepsilon y - V_n) dG(\varepsilon)$ also weakly decreases, as both V_n and $\widehat{\varepsilon}_n^w(y)$ weakly increases. In addition, $\frac{Y_n}{X_n}$ strictly decreases. As a result, the right-hand side of the above equation strictly decreases, while the left-hand side weakly increases. This leads to a contradiction.

Part (i). Suppose y_1^* weakly increases. By Lemma 4 and the result in the last step, V_1 must strictly decrease. As a result, $U_1(y_1^*)$ strictly increases. On the other hand, since $\widehat{\varepsilon}_2(y_1^*) = \widehat{\varepsilon}^m$, $U_2(y_1^*)$ remains the same. This means that a type y_1^* man now strictly prefers class 1 women, and thus y_1^* should strictly decrease. This is a contradiction. Therefore, y_1^* must strictly decrease.

Now consider submarket 2 and suppose y_2^* weakly increases. Since y_1^* strictly decreases, Lemma 4 and the result in the last step imply that V_2 must strictly decrease. Following the same logic as in the previous step, we have a contradiction. Therefore, y_2^* must strictly decrease.

Using the induction as in the previous step, we can show that all y_n^* , $n < N$, must strictly decrease.

Part (ii). The fact that y_n^* strictly decreases directly implies that V_n strictly decreases, as $\widehat{\varepsilon}_{n+1}(y_n^*) = \widehat{\varepsilon}^m$.

Part (iii). Recall that $U_n(y)$ is affected by an increase in X_n only through a change in V_n or $\widehat{\varepsilon}_n(y)$. In addition, we have shown that V_n strictly decreases. The third part of the proposition therefore holds. ■

Proof of Proposition 5.

Proof. We denote the original equilibrium segmentation as \bar{S} , and a new segmentation as \bar{S}' . We first prove the results regarding the equilibrium cutoff types $\{y_j^*\}$, and we prove the claim by ruling out the impossible cases.

Case 1: Suppose $y_n^{*'} \geq y_n^*$ and $y_{n-1}^{*'} \leq y_{n-1}^*$. By the value function (3), $V_n(X_n', \bar{S}) < V_n(X_n, \bar{S}) = V_n$. By Lemma 4, $V_n' = V_n(X_n', \bar{S}') \leq V_n(X_n', \bar{S})$. Therefore, we have $V_n' < V_n$. This implies that $U_n(y_n^*, X_n', \bar{S}') > U_n(y_n^*, X_n, \bar{S})$. Following Corollary 1 and Lemma 6, $y_n^{*'} \geq y_n^*$ means that $V_{n+1}' \geq V_{n+1}$. This implies that $U_{n+1}(y_n^*, X_n', \bar{S}') \leq U_{n+1}(y_n^*, X_n, \bar{S})$. Taking the two inequalities together, we have $U_n(y_n^*, X_n', \bar{S}') > U_{n+1}(y_n^*, X_n', \bar{S}')$. By the property of assortative matching, that means the indifference type $y_n^{*'}$ must satisfy $y_n^{*' < y_n^*$. This is a contradiction.

Case 2: Suppose $y_n^{*'} \geq y_n^*$ and $y_{n-1}^{*'} > y_{n-1}^*$. By Corollary 1 and Lemma 6, $y_{n-1}^{*' > y_{n-1}^*$ implies that $V_{n-1}' < V_{n-1}$. To ensure that $y_{n-1}^{*' > y_{n-1}^*$ is the new marginal type between submarkets $n-1$ and n , it must be the case that $V_n' < V_n$. Following similar steps as in case 1, we get a desired contradiction.

Case 3: Suppose $y_n^{*' < y_n^*$ and $y_{n-1}^{*' < y_{n-1}^*$. By Corollary 1 and Lemma 6, $y_{n-1}^{*' < y_{n-1}^*$ implies that $V_{n-1}' > V_{n-1}$. To ensure that $y_{n-1}^{*' < y_{n-1}^*$ is the new indifference type between submarkets $n-1$ and n , it must be the case that $V_n' > V_n$. Then, using the opposite direction of the proof in case 1, again we can get a contradiction.

By ruling out the above cases, we must have $y_n^{*' < y_n^*$, and $y_{n-1}^{*' \geq \hat{y}_{n-1}^*$.

Part (i). Since $y_n^{*' < y_n^*$, using Corollary 1, we reach the conclusion that $y_i^{*' < y_i^*$ for any $i \geq n+1$. Since $y_n^{*' < y_n^*$, $V_{n+1}' < V_{n+1}$. This again implies that $V_n' < V_n$. The rest of the results follow Corollary 1, and Lemma 6.

Part (ii). Next we show that $y_{n-1}^{*' = y_{n-1}^*$ if $\hat{\varepsilon}_n(y_{n-1}^*) = \hat{\varepsilon}^m$. Since $V_n' < V_n$, $\hat{\varepsilon}_n(y_{n-1}^*) = \hat{\varepsilon}^m$ implies that the new acceptance cutoff $\hat{\varepsilon}'_n(y_{n-1}^*)$ is still $\hat{\varepsilon}^m$. Suppose $y_{n-1}^{*' > y_{n-1}^*$, then $V_{n-1}' < V_{n-1}$, and $U'_{n-1}(y_{n-1}^*) > U_{n-1}(y_{n-1}^*) = U_n(y_{n-1}^*) = U'_n(y_{n-1}^*)$, which contradicts assortative matching as $y_{n-1}^{*' (> y_{n-1}^*)$ is the marginal type. Therefore, we must have $y_{n-1}^{*' = y_{n-1}^*$.

Now we show that $y_{n-1}^{*' > y_{n-1}^*$ if $\hat{\varepsilon}_n(y_{n-1}^*) > \hat{\varepsilon}^m$. Since $V_n' < V_n$, $\hat{\varepsilon}_n(y_{n-1}^*) > \hat{\varepsilon}^m$ implies that the new acceptance cutoff $\hat{\varepsilon}'_n(y_{n-1}^*) < \hat{\varepsilon}_n(y_{n-1}^*)$, which means $U_n(y_{n-1}^*) < U'_n(y_{n-1}^*)$. Suppose $y_{n-1}^{*' = y_{n-1}^*$, then $V_{n-1}' = V_{n-1}$, and $U'_{n-1}(y_{n-1}^*) = U_{n-1}(y_{n-1}^*) = U_n(y_{n-1}^*) < U'_n(y_{n-1}^*)$, which contradicts the presumption that y_{n-1}^* is the indifference type. Therefore, we must have $y_{n-1}^{*' > y_{n-1}^*$.

As to a general $i < n$, the claim can be proved in a similar fashion. The rest of the results follow Corollary 1, and Lemma 6. ■

Proof of Lemma 8.

Proof. Part (i). By (11), for any y with $\hat{\varepsilon}_n(y) = \hat{\varepsilon}^m$, we have

$$\frac{\partial \Gamma(\hat{\varepsilon})}{\partial \gamma} \Big|_{\hat{\varepsilon}=\hat{\varepsilon}^m} \propto 1 + \gamma - \left(1 + \frac{2r}{\alpha}\right) \hat{\varepsilon}^m = 1 + \gamma - \left(1 + \frac{2r}{\alpha}\right) \frac{U_n(y)}{\theta_n y} \equiv T_n(y).$$

By the definition of $T_n(y)$ and the expression of $U_n(y)$, we get

$$\begin{aligned} \theta_n y T_n(y) &= [\theta_n y(1 + \gamma) - U_n(y)] - \frac{1}{\gamma} \int_{\hat{\varepsilon}^m}^{1+\gamma} [\theta_n y \varepsilon - U_n(y)] d\varepsilon \\ &= 2[\theta_n y \frac{1 + \gamma + \hat{\varepsilon}^m}{2} - U_n] - \frac{1 + \gamma - \hat{\varepsilon}^m}{\gamma} [\theta_n y \frac{1 + \gamma + \hat{\varepsilon}^m}{2} - U_n] \\ &= [\theta_n y \frac{1 + \gamma + \hat{\varepsilon}^m}{2} - U_n] \frac{\hat{\varepsilon}^m - (1 - \gamma)}{\gamma} > 0. \end{aligned}$$

Therefore, $T_n(y) > 0$ and $\frac{\partial \Gamma(\hat{\varepsilon})}{\partial \gamma} \Big|_{\hat{\varepsilon}=\hat{\varepsilon}^m} > 0$, which implies that $\hat{\varepsilon}^m$ is increasing in γ .

Part (ii). From the expression of $U_n(y)$ with $\hat{\varepsilon}_n(y) = \hat{\varepsilon}^m$, we take derivative with respect to γ . After using the Envelope Theorem and rearranging, we get

$$\begin{aligned} 2\gamma \left[\frac{r}{\alpha} + 1 - G(\hat{\varepsilon}^m) \right] \frac{\partial U_n(y)}{\partial \gamma} &= [\theta_n y(1 + \gamma) - U_n(y)] - \frac{1}{\gamma} \int_{\hat{\varepsilon}^m}^{1+\gamma} [\theta_n y \varepsilon - U_n(y)] d\varepsilon \\ &= \theta_n y T_n(y) > 0. \end{aligned}$$

Part (iii). By the value function of V_n , we have

$$V_n = \int_{y_n} \frac{\alpha \theta_n y \int_{\hat{\varepsilon}_n(y)}^{1+\gamma} \varepsilon dG(\varepsilon)}{r/q_n + \alpha \int_{y_n} (1 - G(\hat{\varepsilon}_n(y))) dF_n(y)} dF_n(y) \equiv \int_{y_n} Z_n(y) dF_n(y)$$

Let $q_n = 1$. With uniform distribution,

$$Z_n(y) = \frac{\frac{1}{2} \alpha \theta_n y [(1 + \gamma)^2 - \hat{\varepsilon}_n^2(y)]}{2r\gamma + \alpha(1 + \gamma) - \alpha \int_{y_n} \hat{\varepsilon}_n(y) dF_n(y)}.$$

Suppose γ increases to γ' . By part (i), men's acceptance threshold increases to $\hat{\varepsilon}^{m'} > \hat{\varepsilon}^m$. Assume that, under γ' , women in submarket n adopt the same acceptance thresholds as under γ : $\hat{\varepsilon}_n^a(\tilde{y}) = \hat{\varepsilon}_n(\tilde{y})$ for any $\tilde{y} \in \{y_n\}$. Denote $Z_n^a(y)$ as the $Z_n(y)$ under γ' and when women adopt the assumed acceptance threshold. Note that the resulting payoff is not optimal for women under γ' .

First consider high types of y with $\hat{\varepsilon}_n(y) = \hat{\varepsilon}^m$. Among all $\tilde{y} \in y_n$, denote the probability that $\hat{\varepsilon}_n(\tilde{y}) = \hat{\varepsilon}^m$ as \mathbb{P} . Then

$$\begin{aligned} \frac{\partial Z_n^a(y)}{\partial \gamma} &\propto [2(1 + \gamma) - 2\hat{\varepsilon}^m \frac{\partial \hat{\varepsilon}^m}{\partial \gamma}] [2r\gamma + \alpha(1 + \gamma) - \alpha \int_{y_n} \hat{\varepsilon}_n(y) dF_n(y)] \\ &\quad - [(1 + \gamma)^2 - (\hat{\varepsilon}^m)^2] [2r + \alpha - \alpha \mathbb{P} \frac{\partial \hat{\varepsilon}^m}{\partial \gamma}]. \end{aligned}$$

Plug in the optimality condition of $\widehat{\varepsilon}^m$:

$$-2\widehat{\varepsilon}^m[2r\gamma + \alpha(1 + \gamma) - \alpha\widehat{\varepsilon}^m] + \alpha[(1 + \gamma)^2 - (\widehat{\varepsilon}^m)^2] = 0,$$

and we get

$$\begin{aligned} \frac{\partial Z_n^a(y)}{\partial \gamma} &\propto [2r\gamma + \alpha(1 + \gamma) - \alpha\widehat{\varepsilon}^m][(1 + \gamma) - (\frac{2r}{\alpha} + 1)\widehat{\varepsilon}^m] \\ &\quad - [2r\gamma + \alpha(1 + \gamma) - \alpha\widehat{\varepsilon}^m](1 - \mathbb{P})\widehat{\varepsilon}^m \frac{\partial \widehat{\varepsilon}^m}{\partial \gamma} \\ &\quad + \alpha(1 - \mathbb{P})\widehat{\varepsilon}^m(\bar{\widehat{\varepsilon}}_n - \widehat{\varepsilon}^m) \frac{\partial \widehat{\varepsilon}^m}{\partial \gamma}, \end{aligned}$$

where $\bar{\widehat{\varepsilon}}_n$ is the average $\widehat{\varepsilon}_n(y)$ for $\widehat{\varepsilon}_n(y) > \widehat{\varepsilon}^m$. By the optimality of $\widehat{\varepsilon}^m$, we also have

$$[2r\gamma + \alpha(1 + \gamma) - \alpha\widehat{\varepsilon}^m] \frac{\partial \widehat{\varepsilon}^m}{\partial \gamma} = \alpha[(1 + \gamma) - (\frac{2r}{\alpha} + 1)\widehat{\varepsilon}^m].$$

After plugging in $\frac{\partial \widehat{\varepsilon}^m}{\partial \gamma}$ and observing that the last term of $\frac{\partial Z_n^a(y)}{\partial \gamma}$ is always positive, we got

$$\frac{\partial Z_n^a(y)}{\partial \gamma} > 0 \Leftrightarrow 2r\gamma + \alpha(1 + \gamma) - 2\alpha\widehat{\varepsilon}^m \geq 0.$$

This condition is satisfied if $2\widehat{\varepsilon}^m < (1 + \gamma) + \frac{2r\gamma}{\alpha}$.

Next, consider low types of y with $\widehat{\varepsilon}_n(y) > \widehat{\varepsilon}^m$. We can show that

$$\begin{aligned} \frac{\partial Z_n^a(y)}{\partial \gamma} &\propto 2(1 + \gamma)[2r\gamma + \alpha(1 + \gamma) - \alpha \int_{y_n} \widehat{\varepsilon}_n(y) dF_n(y)] \\ &\quad - [(1 + \gamma)^2 - \widehat{\varepsilon}_n^2(y)][2r + \alpha - \alpha\mathbb{P} \frac{\partial \widehat{\varepsilon}^m}{\partial \gamma}] \\ &> [2(1 + \gamma) - 2\widehat{\varepsilon}^m \frac{\partial \widehat{\varepsilon}^m}{\partial \gamma}][2r\gamma + \alpha(1 + \gamma) - \alpha \int_{y_n} \widehat{\varepsilon}_n(y) dF_n(y)] \\ &\quad - [(1 + \gamma)^2 - (\widehat{\varepsilon}^m)^2][2r + \alpha - \alpha\mathbb{P} \frac{\partial \widehat{\varepsilon}^m}{\partial \gamma}] \\ &> 0, \end{aligned}$$

where the first inequality applies because $2r + \alpha - \alpha \frac{\partial \widehat{\varepsilon}^m}{\partial \gamma} > 0$ and the last inequality follows the result in the previous case.

Therefore, when q_n is close enough to 1 and $2\widehat{\varepsilon}^m < (1 + \gamma) + \frac{2r\gamma}{\alpha}$ is satisfied, we have $\frac{\partial V_n}{\partial \gamma} = \int_{y_n} \frac{\partial Z_n(y)}{\partial \gamma} dF_n(y) > 0$.

Part (iv). Recall that $\widehat{\varepsilon}_{n+1}(y_n^*) = \widehat{\varepsilon}^m$. We can explicitly compute how $U_n(y_n^*)$ and $U_{n+1}(y_n^*)$

change as γ increases

$$\begin{aligned} & \frac{\partial U_n(y_n^*)}{\partial \gamma} - \frac{\partial U_{n+1}(y_n^*)}{\partial \gamma} \propto \\ & [\theta_n y_n^*(1 + \gamma) - (1 + \frac{2r}{\alpha})U_n(y_n^*)] - [\theta_{n+1} y_n^*(1 + \gamma) - (1 + \frac{2r}{\alpha})U_{n+1}(y_n^*)] \frac{\frac{r}{\alpha} + 1 - G(\widehat{\varepsilon}_n(y_n^*))}{\frac{r}{\alpha} + 1 - G(\widehat{\varepsilon}^m)} \\ & - [\widehat{\varepsilon}_n(y_n^*) - \frac{U_n(y_n^*)}{\theta_n y_n^*}] \frac{\partial V_n}{\partial \gamma}. \end{aligned} \quad (18)$$

Note that the term of (17) is always positive. This is because $U_n(y_n^*) = U_{n+1}(y_n^*)$, $\theta_n > \theta_{n+1}$, and $G(\widehat{\varepsilon}^m) < G(\widehat{\varepsilon}_n(y_n^*))$. On the other hand, the term of (18) is always negative, as $U_n(y_n^*) < V_n$ and $\frac{\partial V_n}{\partial \gamma} > 0$. Rearranging the above inequality, we have $\frac{\partial U_n(y_n^*)}{\partial \gamma} \geq \frac{\partial U_{n+1}(y_n^*)}{\partial \gamma}$ iff

$$\begin{aligned} & y_n^*(1 + \gamma)(\theta_n - \theta_{n+1}) + \frac{G(\widehat{\varepsilon}_n(y_n^*)) - G(\widehat{\varepsilon}^m)}{\frac{r}{\alpha} + 1 - G(\widehat{\varepsilon}^m)} [\theta_{n+1} y_n^*(1 + \gamma) - U_n(y_n^*)(1 + \frac{2r}{\alpha})] \\ & \geq \frac{\partial V_n}{\partial \gamma} [\widehat{\varepsilon}_n(y_n^*) - \frac{U_n(y_n^*)}{\theta_n y_n^*}]. \end{aligned} \quad (19)$$

Let $\frac{\partial V_n}{\partial \gamma} = x \frac{\partial U_n(y_n^*)}{\partial \gamma}$, and define \widehat{x} as the x such that $\frac{\partial U_n(y_n^*)}{\partial \gamma} = \frac{\partial U_{n+1}(y_n^*)}{\partial \gamma}$. More explicitly, using (19) and the fact that $U_{n+1}(y_n^*) = \theta_{n+1} y_n^* \widehat{\varepsilon}^m$, we can derive

$$\begin{aligned} \widehat{x} &= \frac{[2\gamma \frac{r}{\alpha} + (1 + \gamma) - \widehat{\varepsilon}^m] \frac{(1+\gamma)(\theta_n - 1)}{1+\gamma - (1+\frac{2r}{\alpha})\widehat{\varepsilon}^m} + \widehat{\varepsilon}_n(y_n^*) - \widehat{\varepsilon}^m}{\widehat{\varepsilon}_n(y_n^*) - \frac{\theta_{n+1} \widehat{\varepsilon}^m}{\theta_n}} \\ &= \frac{\theta_n}{\theta_{n+1}} \left\{ 1 + \left(\frac{\theta_n}{\theta_{n+1}} - 1 \right) \frac{(1 + \gamma) \left[1 + \frac{2r}{\alpha} \frac{\gamma + \widehat{\varepsilon}^m}{1 + \gamma - (1 + \frac{2r}{\alpha}) \widehat{\varepsilon}^m} \right] - \widehat{\varepsilon}_n(y_n^*)}{\frac{\theta_n}{\theta_{n+1}} \widehat{\varepsilon}_n(y_n^*) - \widehat{\varepsilon}^m} \right\}. \end{aligned}$$

■

Proof of Proposition 6.

Proof. Part (i). We first show that y_{N-1}^* strictly decreases. Suppose to the contrary, y_{N-1}^* weakly increases; that is $y_{N-1}^{*'} \geq y_{N-1}^*$. This implies that $U'_{N-1}(y_{N-1}^{*'}) - U_{N-1}(y_{N-1}^{*'}) \leq U'_N(y_{N-1}^{*'}) - U_N(y_{N-1}^{*'})$. We also know that (1) $U'_N(y_{N-1}^{*'}) - U_N(y_{N-1}^{*'}) = U_N^a(y_{N-1}^{*'}) - U_N(y_{N-1}^{*'})$ and (2) $U_N^a(y_{N-1}^{*'}) - U_N(y_{N-1}^{*'}) < U_{N-1}^a(y_{N-1}^{*'}) - U_{N-1}(y_{N-1}^{*'})$. Taken together, they imply $U'_N(y_{N-1}^{*'}) - U_N(y_{N-1}^{*'}) < U_{N-1}^a(y_{N-1}^{*'}) - U_{N-1}(y_{N-1}^{*'})$. Therefore, it must be the case that $U_{N-1}^a(y_{N-1}^{*'}) - U_{N-1}(y_{N-1}^{*'}) > U'_{N-1}(y_{N-1}^{*'}) - U_{N-1}(y_{N-1}^{*'})$. This further implies that $V_{N-1}^a < V'_{N-1}$. By Lemma 4, we must have $y_{N-2}^{*'} > y_{N-2}^*$. By the same argument and induction, all cut-off types strictly increase and $V_n^a < V'_n$ for all $n < N$. In particular, $y_1^{*'} > y_1^*$. Recall that $U_1^a(y_1^*) > U_2^a(y_1^*)$. The fact that $y_1^{*'} > y_1^*$ and the single crossing condition imply that $U_1^a(y_1^{*'}) > U_2^a(y_1^{*'})$. By Lemma 4 and the fact $y_1^{*'} > y_1^*$, we reach the conclusion that $U_1'(y_1^{*'}) > U_2'(y_1^{*'})$, which contradicts y_1^{*}' being the new cutoff type.

Second, we show that y_{N-2}^* strictly decreases. Suppose $y_{N-2}^{*'} \geq y_{N-2}^*$. Combined with $y_{N-1}^{*'} < y_{N-1}^*$, Lemma 4 implies that $U'_{N-2}(y_{N-2}^{*'}) - U_{N-2}(y_{N-2}^{*'}) \leq U'_{N-1}(y_{N-2}^{*'}) - U_{N-1}(y_{N-2}^{*'})$. The rest of the proof is similar to that of the previous step. Thus, $y_{N-2}^{*'} < y_{N-2}^*$. By induction, we can show that $y_n^{*'} < y_n^*$ for all n .

Part (ii). Given the conditions, part (ii) of Lemma 8 has shown that $V_1^a > V_1$. By part (i), y_1^* decreases, which by Lemma 4 implies that $V_1' > V_1^a$. Therefore, $V_1' > V_1$.

Part (iii). Consider the highest types of men in any submarket after an increase in γ . Their acceptance cutoff is $\widehat{\varepsilon}^m$ if they are in submarket 1 or if they are in submarket $n > 1$ and the change in γ is small. Then an increase in V_n does not affect their expected payoff. Therefore, their expected payoffs strictly increase as shown earlier in Lemma 8.

Next, consider y_n^* for any $n < N$. Because y_n^* strictly decreases in γ , we have $U'_n(y_n^*) > U'_{n+1}(y_n^*) = U_{n+1}^a(y_n^*) > U_{n+1}(y_n^*) = U_n(y_n^*)$. The first equality follows from the fact that for type y_n^* his acceptance cutoff in submarket $n+1$ is $\widehat{\varepsilon}^m$. This chain of inequalities implies that $U'_n(y_n^*) > U_n(y_n^*)$. According to the continuity, the expected payoffs of the lowest types of men in submarket $n < N$ also increase. ■

Proof of Proposition 7.

Proof. To highlight the importance of α , we will write α as an argument of the relevant functions whenever it is necessary. Given a segmentation $\{y_n\}$ and men's acceptance threshold $\widehat{\varepsilon}^{m'}$, the value function of a type n woman can be written as

$$\begin{aligned} rV'_n(\alpha) &= \alpha q_n \int_{y_n} \max_{\varepsilon_n^w(y)} \left\{ \int_{\max\{\varepsilon_n^w(y), \widehat{\varepsilon}^{m'}\}}^{1+\gamma} [\theta_n y \varepsilon - V'_n(\alpha)] d\widetilde{G}(\varepsilon) \right\} dF_n(y) \\ &= (k\alpha) q_n \int_{y_n} \max_{\varepsilon_n^w(y)} \left\{ \int_{\max\{\varepsilon_n^w(y), \widehat{\varepsilon}^{m'}\}}^{1+\gamma} [\theta_n y \varepsilon - V'_n(\alpha)] dG(\varepsilon) \right\} dF_n(y). \end{aligned}$$

Similarly, given women's threshold $\widehat{\varepsilon}_n^{w'}$, the value function of a type y man in submarket n is

$$\begin{aligned} rU'_n(y; \alpha) &= \alpha \max_{\varepsilon^m} \left\{ \int_{\max\{\widehat{\varepsilon}_n^{w'}(y), \varepsilon^m\}}^{1+\gamma} [\theta_n y \varepsilon - U'_n(y; \alpha)] d\widetilde{G}(\varepsilon) \right\} \\ &= (k\alpha) \max_{\varepsilon^m} \left\{ \int_{\max\{\widehat{\varepsilon}_n^{w'}(y), \varepsilon^m\}}^{1+\gamma} [\theta_n y \varepsilon - U'_n(y; \alpha)] dG(\varepsilon) \right\}. \end{aligned}$$

Combined with the fact that men's acceptance threshold in the baseline model with $k\alpha$ is greater than \underline{x} , the second lines of the above two equations show that the maximization problems of men and women are equivalent to those in the baseline model with $k\alpha > \alpha$. That is, $V'_n(\alpha) = V_n(k\alpha)$ and $U'_n(y; \alpha) = U_n(y; k\alpha)$. ■

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