Search with Private Information: Sorting, Price Formation, and Convergence to Perfect Competition

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Abstract

When there are complementarities in joint production, will decentralized trading with information frictions become efficient as search frictions vanish? We consider a dynamic market where heterogeneous buyers and sellers are randomly paired in each period, with match values being supermodular in both types and production costs depending on seller's type. Within each match, the seller type becomes observable while the buyer type remains private, and the seller makes a take-it-or-leave-it offer. We first highlight sellers' trade-off between a higher price and a higher selling probability under extremely large search frictions when agents completely disregard future payoffs. Log-supermodularity of the gains from trade, which equals the match value minus the production cost, is necessary and sufficient for positive assortative matching (PAM). This means that when the gains from trade is decreasing (increasing) in seller's type, supermodularity is not necessary (not sufficient) for PAM. When search frictions vanish, the condition for PAM returns to supermodularity, as sellers' incentive to secure trade in any given period becomes inconsequential. We then demonstrate that the steady state equilibria exist and converge to the competitive limit as search frictions vanish.

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1. Introduction

How much utility a buyer enjoys from certain goods or services is often determined jointly by the characteristics of the buyer and seller. For example, an online shopper's utility depends on various seller qualities such as product authenticity, speed of delivery, and the shopper's own willingness to pay for these qualities. There are often search and information frictions in the decentralized markets where these goods are traded and hence the search equilibrium is usually inefficient. While we know a lot about the convergence properties when utility only hinges on buyer's type, we do not know as much about the conditions under which search equilibria converge to perfect competitive equilibrium when utility also depends on seller's type. To this end, this paper investigates how goods and services with supermodular match values are traded in a large market with information and search frictions. What are the economic forces that affect the sorting pattern with the presence of frictions? What are the conditions of the match value under which market outcomes, including equilibrium sorting, speed of trade, and division of the surplus, converge to competitive equilibrium as search frictions vanish?

To answer these questions, our model builds on earlier works studying the convergence properties of equilibrium with independent private values. We depart from the literature by assuming that the match value is supermodular in buyer and seller types. In other words, the independent private value would be a special case in which the match value is constant for any seller type. With a supermodular match value, the convergence to competitive equilibrium requires efficient sorting in addition to no delay in trade.

More precisely, in our model, the market opens periodically and runs forever, with some exogenous measure of new buyers (which we will refer to as "he") and sellers (which we will refer to as "she") entering each time the market opens. Buyers and sellers are heterogeneous and have persistent types. They are randomly matched pairwise in each period. In each meeting, the seller's type is jointly observed, but the buyer's type remains private, which is the source of information friction in this model. In turn, the seller has all of the bargaining power and can make a take-it-or-leave-it price offer. If the pair agree to trade, the buyer pays the price and receives the joint output[1] which is supermodular and increasing in both agents' types, the seller gets the price and incurs the production cost that depends on her type, and both of them leave the market permanently. Otherwise, the pair is dissolved and each of them is randomly matched in the next period (given that he or she survives an exogenous exit shock). The exit shock captures the search friction in this model.

[1] We will use "output" and "match value" interchangeably throughout the paper.

We choose the bargaining protocol and the information structure so that the model can be tractable and at the same time, it reflects a broad set of applications. It is straightforward to imagine a market for vertically-differentiated products where consumers differ in preferences. Our model can be seen as the case in which a buyer's preference is his private information. For online shopping, for example, sellers choose prices without knowing how much a buyer values speedy delivery. The buyers in our model can also represent clients who seek services. For instance, sellers in our model can represent the intermediary agencies for domestic helper in Hong Kong[2], providing diverse services at different prices. Some have a pool of more experienced helpers than others, which means that their clients can save some screening effort. A client's characteristics, such as the time costs, together with the qualities of services jointly determine a client's utility. Our model could also draw motivation from the labor market. For consistency, though, we describe the model's agents as buyers and sellers throughout the analysis to follow.

Our first main result concerns the condition of the output function leading to positive sorting. In the model, a seller can only choose one optimal price because of the information friction. Equivalently, she is picking the lowest marginal buyer who is indifferent between buying and not buying.[3] If the seller chooses a higher marginal buyer, she obtains a higher price but the per-period trading probability is lower. In other words, information friction creates a trade-off for the seller between the term of trade and trade probability, both of which depend on the seller's type. When the output function is supermodular, a higher type seller enjoys a larger increase in price by raising the marginal type, which leads to positive assortative matching (PAM). At the same time, a seller's gain from trade equals the price minus the production cost. If it increases in seller type, a higher type seller would also have more to lose from any trade delay, which leads to negative assortative matching (NAM). Then supermodularity is not sufficient for PAM. On the other hand, if the net gain from trade decreases in seller type, then a higher type seller would have weaker incentive to secure trade, which reinforces PAM. In this case, supermodularity is not necessary for PAM.

The above discussion suggests that how production cost changes in a seller's type also

[2]There are over $3,000$ intermediary agencies for domestic helper in Hong Kong. Domestic helpers perform tasks such as cleaning, cooking, taking care of seniors and children for their employers. To get an idea about the size of the market, domestic helpers, mainly from the Philippines and Indonesia, comprise about 5% of the population in Hong Kong. The majority of the Filipino helpers and all of the Indonesian helpers (required by law) are recruited via intermediary agencies.

[3]When trade is hindered by search friction, a seller could trade with buyers whose types lie in several disconnected intervals, as shown in Shimer and Smith (2000). This means that a seller can have multiple marginal buyers. However, once one marginal type is chosen, all the others are uniquely determined in equilibrium, as the seller only has one degree of freedom, which is the price. For the intuition presented here, we therefore assume that the seller chooses the lowest marginal buyer without loss of generality.

affects equilibrium sorting with the presence of search frictions. Moreover, the latter case means that supermodularity of the output function may not be necessary for PAM when there is search friction. The existing literature, to our best knowledge, concludes that stronger complementarity is necessary in a frictional environment. This is because sellers are endowed with their products in other models. Then a higher type seller always has a stronger incentive to secure trade because she has more to lose. But this does not hold true in our model when we incorporate production, as the production cost may increase quickly in a seller's type.

The relevance of the trade-off also depends on how large the search friction is. This dependence results from the interplay between information and search frictions: sellers worry about the probability of trade because of the information friction, and they care more when search frictions are larger. To clearly demonstrate this point, we consider two scenarios with the most drastic comparison: one with the largest search friction in which an agent only has one meeting in a lifetime and the other in which search frictions vanish so the meeting opportunities arrive arbitrarily frequently.[4] In the first scenario, the sorting is positive if and only if the gain from the trade is log-supermodular. Notice that log-supermodularity of the gain does not always require stronger complementarity of the output function as it does with supermodularity. In fact, when the gain decreases in seller's type, a supermodular output function is sufficient but not necessary. In other words, the shape of the production cost function also matters for sorting when search frictions are large. This result matches our earlier intuition. In the second scenario, the time between two periods shrinks to zero and hence sellers' incentive to secure trade in any given period becomes inconsequential. Then the resistance to positive sorting (if any) disappears and supermodularity reemerges as the condition for PAM. This also implies that the shape of the production cost has no impact on sorting in the limit This is because sellers do not care about the trade probability as long as it is positive, which then means that it does not matter how the sellers' gain from trade changes in seller type.

The second main result of this paper is that, as search frictions vanish, search equilibria converge to perfect competitive equilibrium. First of all, the equilibrium sorting is not only positive but also converges to Becker's one-to-one sorting: a buyer who is a percentile of the

[4]We leave the characterization of interior search frictions to future works, as it requires additional effort due to the complexity of equilibrium decisions and interactions. In contrast to models assuming Nash bargaining (e.g., Shimer and Smith (2000), Smith (2006) and Atakan (2006)), the prices arise endogenously in our model and the endogenous division of surplus certainly feeds back to the equilibrium matching sets. Also, comparing to models with independent private values, a buyer's utility directly depends on the seller's type, which means that two sellers with different types will have different trading probabilities even if they ask for the same price. This greatly complicates the problem. Moreover, an agent's expected payoff depends on all of the other agents' strategies as well as the endogenous type distributions.

buyer-entrant's type distribution only trades with sellers whose types are arbitrarily close to the same percentile of the seller-entrant's type distribution. Intuitively, as the length of each period shrinks to zero, a seller becomes pickier and chooses a price that is only accepted by buyers whose types are arbitrarily close to a "targeted type." This "targeted type" strictly increases in the seller's type, as supermodularity means that a seller with a higher type has a lower cost of providing a certain level of utility to a buyer. Secondly, there is no trade delay in the limit. The convergence to one-to-one sorting implies that the per-period trading probabilities for all agents diminish to zero. The expected time that it takes to trade therefore does not necessarily reduce to zero even if the trading opportunities appear arbitrarily quickly. In equilibrium, however, we can show that the time between two periods converges to zero faster than the per-period trading probability does. Therefore, any agent indeed expects to trade immediately after entry. In addition, the equilibrium prices also converge to competitive prices. The monopolistic aspect of bilateral trade is weakened as the time preference of agents is removed in the limit. As a result, the division of the surplus becomes such that agents appropriate their marginal contributions.

The paper proceeds as follows: Section 2 frames our findings in the context of related literature. Section 3 introduces the model, lays out the equilibrium conditions and formally defines search equilibrium. Section 4 characterizes the sorting and price formation when the model is effectively static. Section 5 offers results for the other extreme case (as search frictions vanish) and shows that search equilibria exist and converge to perfect competition. We also discuss cases where search equilibria fail to converge to the competitive limit when there is no information friction or when exits are replaced with clones. Finally, section 6 concludes the paper.

2. Related Literature

Our paper is related to the theory of search and matching. A large number of papers have investigated whether search equilibria converge to the competitive limit as search frictions vanish. The convergence result holds in settings with perfect information (e.g., Rubinstein and Wolinsky (1985) (1990), Gale (1986) (1987), Mortensen and Wright (2002)) and settings with private information (e.g., Satterthwaite and Shenyerov (2007) (2008), Atakan (2009), Shneyerov and Wong (2010), Lauermann (2013)). The main difference between our model and the others is that the match value here also depends on the seller's type. In previous works assuming independent private value, a buyer's purchasing decision only hinges on the price. Convergence then places restrictions on the bargaining protocols, information structure and how entrants replace exits. When the match value is supermodular, we need further conditions on the match value and production cost. To restrict attention to these new conditions, we choose the bargaining protocol and other elements in the model as will be specified in the next section. We already know from the existing works that search equilibria converge to perfect competitive equilibrium under these assumptions when the match value is constant in seller's type.

Our analysis also builds upon previous theoretical works related to assortative matching. A standard benchmark is the frictionless "Walrasian" setting studied by Becker (1973) and Rosen (1974), where there is full information regarding prices and types, and meeting trading partners is fully costless. Becker famously demonstrated that supermodular production functions give rise to PAM. Beyond this, though, recent studies have taken a renewed interest in understanding how sorting is impacted by departures from the frictionless benchmark. Among these frictional extensions, the setting we study is especially well-suited to compare settings involving two particular classes of frictions—the bilateral monopoly which arises in random search and the coordination friction in directed search. We elaborate upon these connections below.

As discussed in the introduction, one common conclusion in the existing literature on both random search and directed search is that because sellers care about trade probability with search frictions, stronger complementarity than supermodularity is necessary for positive sorting. We point out that when there is production, the concern about trade probability places a condition on the gain from trade, which depends on both the match value and production cost. This means that the regularity condition on the match value is related to the shape of the production cost function. If the production cost increases in seller type rapidly, the supermodularity of the match value is actually sufficient but not necessary for PAM with extremely large search frictions.

There are obvious connections between our study and a series of papers on sorting with random search and transferable utility (e.g., Shimer and Smith (2000) and Atakan (2006)). Our study differs from these primarily in its incorporation of buyer's private information—in the studies mentioned above, there is full information and an exogenously given sharing rule in each meeting of potential partners. Moreover, Shimer and Smith (2000) focus more on providing sufficient conditions for sorting (log-supermodularity of the match value) given some arbitrary level of search friction. In this paper, we focus on how sorting conditions vary for different search friction magnitudes. We identify the source of resistance to sorting as information friction in the static case, and extend this intuition to explain why the sorting condition is weakened as search frictions vanish. In addition, compared to Shimer and Smith (2000), we are also particularly interested in investigating if and when the search equilibria converge to perfect competition.

There are also papers establishing relationships between coordination friction and equilibrium sorting. Eeckhout and Kircher (2010) study a one-shot setting with buyer's private information and directed search. The coordination friction is governed by the search technology, i.e., a function which maps the buyer-seller composition at each location to a realized number/measure of bilateral matches. The authors show that stronger complementarity is required when it is harder for a seller to successfully meet a buyer given the buyer-seller ratio. Note that, when viewed through the lens of time, our random search and their directed search setting appear to be much more closely related. Eeckhout and Kircher's sellers use posted prices to sort agents prior to meeting (*ex ante* sorting), while our sellers sort only after the buyer has arrived (*ex post* sorting). Intuitively, we focus on settings in which the search process is too imprecise for agents to be guided toward specific trading partners. In some sense, our analysis can be viewed as a bridge between studies of private information in directed search and studies considering full information and random search.

3. The Model

3.1 Basic Environment

We consider a discrete-time, infinite horizon random search model with heterogeneous buyers and sellers. We focus on steady state equilibrium throughout the paper and hence the time indexes are omitted for all the variables. At the beginning of each period, there is a continuum of new buyers and sellers entering the market, with measures normalized to 1. Their types are randomly drawn from distributions with pdfs $\gamma_B(x)$ for buyers and $\gamma_S(y)$ for sellers over the bounded intervals $X = [\underline{x}, \overline{x}] \subset \mathbb{R}_+$ and $Y = [\underline{y}, \overline{y}] \subset \mathbb{R}_+$, assuming $\underline{x} > 0$ and $\underline{y} > 0$. The market participants of one period include the new buyers, sellers and the incumbents. The pdf of the buyer and seller type distribution in the market are denoted as $f_B(x)$ and $f_S(y)$, respectively.

Each buyer is randomly matched with one seller (and vice versa) in each period. In each pair, the seller type becomes observable, while the buyer type remains private. Sellers, however, have the power to make take-it-or-leave-it offers $P(y)$ to buyers.[5] Therefore, a type

[5] According to Riley and Zeckhauser (1983), if a seller can commit to a selling mechanism when facing a

y seller's strategy is the price offer and a type x buyer's strategy is the set of sellers with whom he is willing to trade. If trade occurs, output $z(x, y)$ is produced at cost $c(y) \geq 0$ and both parties leave the market permanently with utility $z(x, y) - P(y)$ for the buyer and $P(y) - c(y)$ for the seller. Those who do not trade experience an exogenous exit shock with probability $1 - \beta$, in which case they leave the market. Otherwise, remaining buyers and sellers play the same game in the next period. Agents discount future payoffs only because of the exit shock, so the relevant discount factor for all agents is β. We assume that when choosing their strategies, agents know the market condition in the sense that buyers know $f_S(y)$ and $P(y)$ and sellers know $f_B(x)$ and $V(x)$, which is the equilibrium payoff of a buyer of type x, for any x and y on the support.

For subsequent analysis, we impose the following assumptions on $z(x, y)$, $c(y)$, $\gamma_B(x)$ and $\gamma_S(y)$:

Assumption 1: *Over the domain $X \times Y$, the output function $z(x, y)$ is bounded, strictly increasing in x and y and twice continuously differentiable. In addition, $z(x, y) - c(y)$ is*

1. *strictly positive and bounded;*

2. *continuously differentiable and has uniformly bounded first partial derivatives; and*

3. *log-concave in x.*

Notice that we allow $z(x, y) - c(y)$ to be decreasing in y for some range of y. That is, the production cost can increase faster than the output function does. We will show in later section that the condition for positive sorting on the match value is weaker when $z(x, y) - c(y)$ decreases in y.

Assumption 2: *$\gamma_B(x)$ and $\gamma_B(x)$ are continuous, strictly positive and bounded over the domain of X and Y, respectively.*

3.2 Value Functions and Steady State Conditions

Buyer's Problem

Given $\{P(y)\}_{y \in Y}$, a buyer of type x chooses a set of sellers with whom he is willing to trade. Unlike in a frictionless market, equilibria do not entail a deterministic, one-to-one matching. Rather, the buyer trades probabilistically with a seller whose type is randomly

buyer with an unknown type, the optimal mechanism is equivalent to a take-it-or-leave-it offer. Therefore, it is without loss of generality to assume that sellers make such price offers.

drawn from a range of acceptable types. If he trades with a type y seller, the payoff is $z(x, y) - P(y)$. Otherwise, he expects to get some payoff in the next period, denoted as $V(x)$, which is discounted with factor β because of the exogenous exit shock. As a result, a type x buyer is willing to trade with a type y seller if and only if $z(x, y) - P(y) \geq \beta V(x)$.

We therefore denote a type x buyer's surplus from trading with a type y seller as $s(x, y) = z(x, y) - \beta V(x) - P(y)$. If a type y seller is acceptable by a type x buyer, then we say that y is in buyer x's *matching set*. Denote the equilibrium matching set as $\mathbb{M}_B(x)$, which is defined as

$$\mathbb{M}_B(x) = \{y : s(x, y) \geq 0\} \tag{1}$$

A buyer of type x trades if he meets a seller from his matching set. His value function can therefore be expressed as

$$V(x) = \int_{\mathbb{M}_B(x)} (z(x, y) - P(y))dF_S(y) + \left(1 - \int_{\mathbb{M}_B(x)} dF_S(y)\right) \beta V(x)$$

where $F_S(y)$ is the cdf of the seller's type distribution in steady state. Rearrange and we obtain

$$V(x) = \frac{\int_{\mathbb{M}_B(x)} (z(x, y) - P(y))dF_S(y)}{1 - \beta + \beta \int_{\mathbb{M}_B(x)} dF_S(y)}$$

Seller's Problem

In turn, $\mathbb{M}_S(y)$ corresponds to the *matching set* of a type y seller, which is defined as

$$\mathbb{M}_S(y) = \{x : s(x, y) \geq 0\} \tag{2}$$

Obviously, $y \in \mathbb{M}_B(x)$ if and only if $x \in \mathbb{M}_S(y)$. To conveniently identify whether a pair of agents belong to each other's matching set, we sometimes also use the indicator function $d(x, y)$ defined as follows: $d(x, y) = 1$ if and only if $s(x, y) \geq 0$ and $d(x, y) = 0$ otherwise.

By choosing different prices, a seller changes the matching set, as the surplus $s(x, y) = z(x, y) - \beta V(x) - P$ strictly decreases in P. Therefore, given $V(x)$, a type y seller's matching set shrinks as the price increases, implying a lower probability of trade in one period. In other words, a seller faces the trade-off between the term of trade and the probability of trade.

Denote the value function of a type y seller as $\Pi(y)$. Given $V(x)$ and $F_B(x)$, which is the cdf of the buyer's type distribution in steady state, $\Pi(y)$ can be expressed as

$$\Pi(y) = \max_P \{(P - c(y)) \int_{\mathbb{M}_S(y;P,V(x))} dF_B(x) + (1 - \int_{\mathbb{M}_S(y;P,V(x))} dF_B(x))\beta\Pi(y)\} \tag{3}$$

Steady State Condition

The last equilibrium condition is the steady state condition: the outflow of any type must equal the inflow of the same type. The inflow is governed by the entrant type distributions γ_B and γ_S. The outflow of type x buyers consists of two groups. A buyer will exit if he is paired with a seller in his matching set. Otherwise, a buyer would leave the market because of the exit shock. The same accounting applies to the seller side.

The pdfs of the steady state type distributions (f_B, f_S) and pdfs of the entrant type distributions (γ_B, γ_S) therefore must satisfy the two inflow-outflow equations below:

$$\begin{aligned} \hat{f}_B(x) &= \frac{\gamma_B(x)}{(1-\beta) + \beta \frac{\int_{\mathbb{M}_B(x)} \hat{f}_S(y)dy}{\int \hat{f}_S(y)dy}} \\ \hat{f}_S(y) &= \frac{\gamma_S(y)}{(1-\beta) + \beta \frac{\int_{\mathbb{M}_S(y)} \hat{f}_B(x)dx}{\int \hat{f}_B(x)dx}} \\ \text{and } f_B(x) &= \frac{\hat{f}_B(x)}{\int \hat{f}_B(\tilde{x})d\tilde{x}},\ f_S(y) = \frac{\hat{f}_S(y)}{\int \hat{f}_S(\tilde{y})d\tilde{y}} \end{aligned} \tag{4}$$

A steady state search equilibrium is defined formally as follows.

Definition 1: *A search equilibrium consists of buyers' matching set $\mathbb{M}_B(x)$, sellers' matching set $\mathbb{M}_S(y)$, prices $P(y)$, and the pdfs of the steady state distributions $f_B(x)$, $f_S(y)$ such that*

1. *$\mathbb{M}_B(x)$ and $\mathbb{M}_S(y)$ satisfies (1) and (2) respectively;*
2. *$P(y)$ maximizes seller y's expected profit as in equation (3) for any y;*
3. *$f_B(x)$ and $f_S(y)$ are jointly determined by (4).*

Characterizing the equilibrium is not easy, as the buyers' and sellers' decisions and the steady state distributions are related. What is more, the determination of matching sets is more difficult than in settings assuming supermodular match value and Nash bargaining, because the division of the surplus here is endogenously chosen by the sellers. The sellers' maximization problem is also more complicated than in settings assuming independent private values and sellers proposing prices. The fact that match values depend on a seller's type means that two sellers charging the same price have different trading probabilities if their types differ. Then how the equilibrium price would change in a seller's type would also depend on how the trading probability changes in price for different types of sellers, which itself is an equilibrium object.

We therefore restrict our attention to two cases. We first investigate the scenario with extremely large search frictions. That is, agents completely disregard the continuation payoffs. In this case, a seller's maximization problem is simple and the steady state distribution mechanically equals the entrants' type distribution. Next, we look into the limiting case where the search frictions vanish. In this case, extremely patient sellers do not care about the per-period trading probability as long as they can almost surely trade before the exogenous exit shock.

4. One-Shot Bilateral Monopoly: $\beta = 0$

In this section, we consider the case in which all of the agents face terminal shock after one period, which is essentially a one-shot bilateral monopoly. In other words, the search frictions are at the maximal level and agents do not value the future at all ($\beta = 0$).

4.1 Equilibrium Matching Set

In this case, the surplus function $s(x, y)$ equals $z(x, y) - P(y)$, which is strictly increasing in x. If a seller of type y sets the price $P(y) = z(x, y)$, then any buyer with a type above x is willing to accept the offer. Therefore, choosing the optimal price $P(y)$ is equivalent to selecting the marginal type $x^*(y)$ to maximize the expected profit. That is,

$$\Pi(y) = \max_{\tilde{x}}\{[z(\tilde{x}, y) - c(y)](1 - \Gamma_B(\tilde{x}))\}$$

Here $F_B(x) = \Gamma_B(x)$, since all buyers exit the market after one period.

Assumption 3: *(i)* $\frac{\gamma_B(x)}{1-\Gamma_B(x)}$ *is strictly increasing in* x *for any* $x \in [\underline{x}, \bar{x}]$*. (ii)* $\max_y\{\frac{z_1(\underline{x},y)}{z(\underline{x},y)-c(y)}\} > \gamma_B(\underline{x})$.

To make sure that $x^*(y)$ is unique, we assume that $\Gamma_B(x)$ has strictly increasing hazard rate.[6] In addition, we exclude the trivial cases where $x^*(y) = \underline{x}$ for any y by assuming part (ii) of the above assumption.

[6] The assumption can be weakened if $z(x, y) - c(y)$ is strictly log-concave in x. In that case, $\frac{\gamma_B(x)}{1-\Gamma_B(x)}$ only needs to be weakly increasing in x.

Proposition 1: *When $\beta = 0$, a type y seller chooses equilibrium price $P(y) = z(x^*(y), y)$ and trades with any buyer whose type is above $x^*(y)$, where $x^*(y)$ is uniquely determined by*

$$
\begin{aligned}
&x^*(y) = \underline{x}, \text{ if } z_1(\underline{x}, y)(1 - \Gamma_B(\underline{x})) \leq [z(\underline{x}, y) - c(y)])\gamma_B(\underline{x}) \\
&z_1(x^*(y), y)(1 - \Gamma_B(x^*(y))) = [z(x^*(y), y) - c(y)]\gamma_B(x^*(y)), \text{ otherwise.}
\end{aligned} \tag{5}
$$

Both $P(y)$ and $x^*(y)$ are continuous in y.

Proof: Condition (5) is obtained by taking the first order condition of the seller's objective function. To show the uniqueness of $x^*(y)$, for any given y, consider the following two functions in x,

$$
\begin{aligned}
L(x; y) &= \frac{z_1(x, y)}{z(x, y) - c(y)} \\
R(x) &= \frac{\gamma_B(x)}{1 - \Gamma_B(x)}
\end{aligned}
$$

The function $L(x; y)$ weakly decreases in x, as implied by the log-concavity of $z(x, y) - c(y)$ in x. The function $R(x)$ strictly increases in x, following the assumption of strictly increasing hazard rate. Both $L(x; y)$ and $R(x)$ are continuous in x. Therefore, they can intersect at most once.

First note $x^*(y) < \bar{x}$ for any y, because $R(\bar{x})$ is always larger than $L(\bar{x}; y)$. If $R(\underline{x}) < L(\underline{x}; y)$, then the equation $L(x; y) = R(x)$ has a unique interior solution. We can rearrange the equation and obtain the second case in condition (5). If $R(\underline{x}) \geq L(\underline{x}; y)$, then $R(x) > L(x; y)$ for any $x \in (\underline{x}, \bar{x}]$. This corresponds to the first case in condition (5).

The optimal price has been shown to equal to $z(x^*(y), y)$ in the preceding discussion. The continuity of $x^*(y)$ and $P(y)$ follows from the fact that $L(x; y)$ is continuous in y. ■

As usual, an (interior) x^* is chosen so that the marginal revenue of increasing x^* equals the marginal cost. The left-hand side of equation (5) represents the marginal revenue. The resulting price increment is $z_1(x^*(y), y)$ and the seller can collect this increment when trade happens with the probability of $1 - \Gamma_B(x^*(y))$. The right-hand side is the marginal cost of increasing the marginal buyer type. The seller can no longer sell to the buyers of type $x^*(y)$. The resulting loss equals the net gain, $z(x^*(y), y) - c(y)$, times the probability of meeting a buyer of type $x^*(y)$, $\gamma_B(x^*(y))$.

4.2 Condition for Positive Assortative Matching

Let us now characterize sorting. The matching sets in an environment with search frictions are normally non-degenerate because it takes time for a buyer to meet a seller. With a degenerate matching set, the probability of trade within one period is zero and the buyer would almost surely experience the exit shock before any trade takes place. As a result, we can no longer use Becker (1973)'s definition of PAM. Instead, we follow Shimer and Smith (2000)'s definition: the set of mutually agreeable matches must form a lattice for PAM to occur. More explicitly:

Definition 2: *Take $x_1 < x_2$ and $y_1 < y_2$. There is positive assortative matching (PAM) if $y_1 \in \mathbb{M}_B(x_1)$ and $y_2 \in \mathbb{M}_B(x_2)$ whenever $y_1 \in \mathbb{M}_B(x_2)$ and $y_2 \in \mathbb{M}_B(x_1)$.*

Under the threshold rule and the assumptions that ensure differentiability, the definition of PAM amounts to the marginal type being non-decreasing in y. That is, sorting is positive if and only if $\frac{\partial x^*(y)}{\partial y} \geq 0$.

Proposition 2: *When $\beta = 0$, sorting is positive given any permissible $\Gamma_B(x)$ if and only if the net gain from trade $z(x,y) - c(y)$ is log-supermodular.*

Proof: If $x^*(y)$ is on the boundary for some y, then $\frac{\partial x^*(y)}{\partial y} = 0$. The matching is assortative.

If $x^*(y)$ is an interior solution, then from (5),

$$\begin{aligned}\frac{\partial x^*(y)}{\partial y} &= \frac{z_{12}[1-\Gamma_B(x^*)] - [z_2 - c'(y)]\gamma_B(x^*)}{2z_1\gamma_B(x^*) + (z-c(y))\gamma_B'(x^*) - z_{11}[1-\Gamma_B(x^*)]} \\ &= \frac{z_{12}(z-c(y)) - z_1(z_2 - c'(y))}{(z_1)^2} \times \frac{1}{2 + \frac{(z-c(y))}{z_1}\frac{\gamma_B'}{\gamma_B} - \frac{z_{11}(z-c(y))}{(z_1)^2}}\end{aligned} \tag{6}$$

The second equality follows after plugging in the first order condition and rearranging terms.

We also know that $\frac{1-\Gamma_B}{\gamma_B}$ strictly decreases in x

$$\begin{aligned}&\frac{\partial}{\partial x}\left(\frac{1-\Gamma_B}{\gamma_B}\right) = \frac{-\gamma_B^2 - (1-\Gamma_B)\gamma_B'}{\gamma_B^2} < 0 \\ &\Rightarrow 1 + \frac{1-\Gamma_B}{\gamma_B}\frac{\gamma_B'}{\gamma_B} > 0 \Rightarrow \frac{z-c(y)}{z_1}\frac{\gamma_B'}{\gamma_B} > -1\end{aligned}$$

Therefore,

$$2 + \frac{(z-c(y))}{z_1}\frac{\gamma_B'}{\gamma_B} - \frac{z_{11}(z-c(y))}{(z_1)^2} > 2 - 1 - 1 = 0$$

The sorting is positive given any type distribution if and only if $z_{12}(z - c(y)) - z_1(z_2 - c'(y)) \geq 0$, i.e., $z(x, y) - c(y)$ is log-supermodular. ■

The above proposition establishes that with the maximum search friction, we need a log-supermodular gain from trade $(z(x, y) - c(y))$ to ensure positive sorting. When we compare this to the sorting conditions obtained in earlier works, it suggests that PAM requires a different degree of match value complementarity depending on the shape of the production cost. Whether a supermodular $z(x, y)$ implies a log-supermodular $z(x, y) - c(y)$ depends on how the gain changes in seller type. The following proposition is derived directly from proposition 2 and hence the proof is skipped.

Proposition 3: *When $z_2(x, y) - c'(y) > 0$ for any x, the supermodularity of $z(x, y)$ is necessary but not sufficient for PAM. When $z_2(x, y) - c'(y) < 0$ for any x, the supermodularity of $z(x, y)$ is sufficient but not necessary for PAM.*

Previous works have pointed out that PAM in general requires stronger complementarity than supermodularity when the search is frictional, assuming that sellers are endowed with their products.[7] We show that this is true only when the match value increases faster in y than the production costs does. If the opposite holds, that is, when the production costs increases faster in y than the match value does, then the equilibrium sorting may be positive even with a submodular output function.[8]

The intuition behind Proposition 2 and 3 is a key to our argument. It also helps us understand the sufficiency of supermodularity when search frictions vanish in the next section. Consider two sellers with types y_1 and y_2 ($y_1 < y_2$), respectively. Imagine that the two sellers are currently choosing the same marginal buyer type x, which is seller 1's optimal marginal type. Consider the trade-offs for each seller in terms of price and trade probability associated with slightly increasing the marginal type.

[7]For example, Eeckhout and Kircher (2010) show that the sorting is positive if and only if the output function is n-root-supermodular.

[8]One may ask the following question: when the product cost increases faster in y than the match value does, why would not a seller with a higher y switch to the production technology with a lower y? An implicit assumption in our model is that a seller is endowed with technology which she cannot change. So such switching is not possible. In reality, the switching is also difficult if not completely impossible in many situations. Take the wine industry as an example. A winery can choose between traditional winemaking processes, which produces higher quality wine at higher costs, and modern processes, which scarifies quality to save costs. The winery is facing many constraints when making this decision, such as orographic conditions and regulations. For instance, it is difficult to use tractors to pick grapes planted on a hill. Also, in the Champagne Region of France, the production of champagne must follow the traditional method for the wine to be labelled as champagne.

The marginal benefit of increasing the marginal type equals the price increment times the trading probability, i.e., $z_1(x,y)[1-F_B(x)]$. If the output function is supermodular, the price increment $z_1(x,y)$ is greater for the high type seller 2. The two sellers are currently choosing the same marginal buyer type. So their trading probabilities are the same. Overall, a higher type seller enjoys a higher marginal benefit given supermodularity and hence has stronger incentives to increase his marginal buyer type, suggesting positive sorting.

The marginal cost of increasing the marginal type equals the reduction in trading probability times the net gain, i.e., $(z(x,y)-c(y))f_B(x)$. The two sellers experience the same level of trading probability reduction. Which seller has a higher marginal cost, as a result, depends on who has a larger gain from trade. If seller 2 (the higher type seller) has a larger gain, then she loses more from the same reduction in trading probability. Hence, a seller of a higher type also has a higher marginal cost and thus a stronger incentive to secure trade through lowering her marginal buyer type. This suggests negative sorting. In this case, supermodularity is clearly insufficient for PAM. On the other hand, if seller 1 (the lower type seller) has a larger gain, then she has a higher marginal cost of increasing the marginal type. Recall that seller 1 also has a lower marginal benefit if the output function is supermodular. This means that the equilibrium sorting can be positive even when the match value has weaker complementarity than supermodularity.

To further illustrate the role of private information, let us compare the current sorting result with the one in Shimer and Smith (2000). In their setting, agents' types become observable after they are paired. In the static case, an agent is willing to trade with any type, regardless of the bargaining power allocation. Based on the definition, matching is always positive with any output function. The difference results from the fact that Shimer and Smith's sellers do not face the trade-off between price and trading probability, because there is no information friction. When a seller in their model raises the marginal buyer type, there is no marginal benefit (as the prices charged to other buyers remain the same) but only marginal cost (as the trading probability is lower).

Moreover, we can also characterize how prices change in seller types. We find that prices increase in seller types if the output function exhibits a sufficiently strong combination of supermodularity and log-concavity in buyer type.

Corollary 1: *$P(y)$ increases in y if $z(x,y)$ is concave in x and supermodular, or if $z(x,y)-c(y)$ is log-concave in x and log-supermodular.*

Proof: We know from the preceding analysis that

$$\begin{aligned}\frac{\partial P(y)}{\partial y} &= z_1\frac{\partial x^*}{\partial y} + z_2 \\ &\propto z_{12}(z-c) - z_1(z_2 - c') + 2z_1z_2 + z_1z_2\frac{z-c}{z_1}\frac{\gamma'}{\gamma} - z_1z_2\frac{z_{11}(z-c)}{(z_1)^2} \\ &> z_{12}(z-c) - z_1(z_2-c') + z_1z_2[1 - \frac{z_{11}(z-c)}{(z_1)^2}]\end{aligned}$$

The last inequality follows from the result $\frac{z-c(y)}{z_1}\frac{\gamma_B'}{\gamma_B} > -1$ obtained in the proof of Proposition 2. Therefore, $\frac{\partial P(y)}{\partial y} > 0$ if $z_{12} \geq 0$ and $z_{11} \leq 0$, or if $\frac{z_{11}(z-c)}{(z_1)^2} \leq 1$ and $z_{12}(z-c) - z_1(z_2 - c') \geq 0$. ■

5. Frictionless Limit: $\beta \to 1$

In this section, we examine the equilibrium outcomes when the time between two consecutive periods shrinks to zero—that is, as the discount factor β approaches 1. To be more precise, we denote all agents' common discount rate as $r > 0$ and the time between two periods as t. Then the discount factor is $\beta(t) = e^{-rt}$. As $t \to 0$, $\beta(t)$ converges to 1. Whenever it causes no confusion, we omit t and directly say that $\beta \to 1$. The measure of entrants per period is also scaled accordingly, which equals to $1/t$, because the per-unit time measure is normalized to one.

5.1 Sorting Converges to One-to-One PAM

When there is neither search friction nor information friction, it is well-known that the equilibrium matching is one-to-one positive assortative, which is formally defined below.

Definition 3: *There is one-to-one PAM if there exists a strictly increasing and continuous function $m(x)$ defined on $[\underline{x}, \bar{x}]$ with $m(\underline{x}) = \underline{y}$ and $m(\bar{x}) = \bar{y}$, such that $\mathbb{M}_B(x) = m(x)$.*

It is straightforward to verify that one-to-one PAM implies PAM.

As search frictions vanish, we find that one-to-one PAM can be restored even if information frictions remain. That is, a supermodular $z(x, y)$ is sufficient to ensure convergence to one-to-one PAM. Write $\mu(\alpha) = \int_{\tilde{x}\in\alpha} dF(\tilde{x})$, where $F(\tilde{x})$ is the cdf of variable $\tilde{x}$.

Proposition 4: *When $z(x, y)$ is supermodular, for any $\xi > 0$, there exists an $\epsilon > 0$ such that for any $\beta > 1 - \epsilon$,*

1. *$d(x, y) = 1$ if and only if $s(x, y) \in [0, \xi)$;*

2. *$\mu(\mathbb{M}_B(x)) \in [0, \xi)$ and $\mu(\mathbb{M}_S(y)) \in [0, \xi)$;*

3. *the matching sets converge to one-to-one PAM, i.e., there exists a strictly increasing and continuous function $m(x)$ defined on $[\underline{x}, \bar{x}]$ such that (i) $m(\underline{x}) = \underline{y}$, $m(\bar{x}) = \bar{y}$, and (ii) for any (x, y) with $d(x, y) = 1$, $\mid x - m^{-1}(y) \mid < \xi$ and $\mid y - m(x) \mid < \xi$.*

To understand this result, note that sellers still face the trade-off between the price and trading probability in each period, but they care less and less about the latter as they meet buyers more and more frequently. Even with a small matching set, a seller can almost surely sell before experiencing the terminal shock. Thus, a seller has an incentive to raise her price so that she only trades with buyers whose types belong to a small neighbourhood of her most preferred and feasible type.

This targeted type must increase in the seller's type. A seller would like to trade with higher type buyers, ceteris paribus, as the match value is higher. Unfortunately, to attract higher type buyers with the presence of private information, a seller would have to provide them with higher utilities. With supermodular output functions, a higher type seller has a lower cost to provide utilities to buyers.

Notice that the convergence puts no further restriction on the production cost function $c(y)$. The reason is that, as sellers care less and less about the trading probability per period, positive sorting eventually only requires the match surplus $z(x, y) - c(y)$ to be supermodular, which reduces to the supermodularity of $z(x, y)$.

The one-to-one positive assortative matching also requires all buyers, including those with low types, to have the chance to trade. Note that this is not generally true for the static benchmark, but it will indeed hold in the limit. Suppose some lower type buyers are not in any seller's matching set when search friction vanish. These buyers exit the market only when they experience death shock, while higher type buyers exit when they experience death shock as well as when they trade. With slower exit rates, these lower type buyers congest the market. As a result, some sellers will find it profitable to lower their prices slightly to gain a big increase in trading probability.

This convergence to one-to-one PAM may seem to be the only possible result. But it does depend on the specifications of the model. The following two examples show that the equilibrium sorting does not converge to one-to-one matching if we impose alternative assumptions. They also help identify the role of information friction and exogenous entry in determining the limiting sorting pattern.

Example 1: Perfect Information. In the static case, we have established that it is the information friction that impedes sorting. As search frictions diminish, this effect also vanishes, as the next meeting becomes immediate. In this example, we show that in fact some private information is necessary for the convergence to one-to-one sorting if buyers have no bargaining power.

Consider the same environment as specified in the model except for one difference: a buyer's type is observable to the paired seller. Because of infinite rounds of discounting, all buyers have zero continuation payoff. The matching set must then be characterized by a cut-off rule. That is, a seller trades with any buyer whose type is above a certain threshold. Therefore, the matching is never one-to-one and the search equilibrium fails to converge to perfect competition.

This example suggests that convergence requires some underlying factors to ensure a buyer's positive payoff. It could be the buyer's private information as our model, or it might be some positive bargaining power that the buyer has. We will leave the general characterization of the sufficient conditions for convergence (in terms of information set, bargaining protocols, etc.) to future works.

Example 2: Exits are Replaced with Clones. Instead of having a fixed measure of entrants with exogenously given type distributions, suppose agents who exit are replaced by their clones. Then the exogenously given type distributions of incumbents are preserved over time. In this case, some low type buyers may never trade and the matching in the limit does not become one-to-one positive assortative in general. The reason is that even if some lower type buyers never trade, their population still only comprises an exogenously given fraction of the market size and will not grow. Then sellers do not have the incentive to trade with these low type buyers in equilibrium.

5.2 Convergence to Perfect Competition

We have established in the previous section that—as in the Walrasian benchmark–one-to-one PAM requires a supermodular match value as frictions vanish. We show in this section that the equilibrium price function and the equilibrium matching governed by $m(x)$ also converge to their competitive limits. This means that the search equilibrium converges to Walrasian equilibrium as search frictions vanish if the output function is supermodular.

Proposition 5: *When $z(x, y)$ is supermodular, equilibrium prices converge pointwise to the* *price function* $P^*(y)$, *where*

$$P^*(y) = z(\underline{x}, \underline{y}) + \int_{\underline{y}}^{y} z_2(m^{-1}(\tilde{y}), \tilde{y}) d\tilde{y}$$

Under the price schedule $P^*(y)$, each agent appropriates the marginal contribution calculated given the matching $m(x)$. This is the way surplus is divided in the Walrasian equilibrium. The above proposition shows that the equilibrium price approaches its Walrasian counterpart. Intuitively, because the buyer in a match can meet another seller almost immediately, the seller in the match faces almost perfect competition from other sellers. Thus, in the limit, the price increment of each seller type approaches that type's marginal contribution to the output.

The earlier results of the convergence to one-to-one PAM and equilibrium price are not sufficient for convergence to the Walrasian equilibrium. What still needs to be shown is that the $m(x)$ function, which governs the sorting pattern, converges to the competitive limit. In addition, the convergence requires no delay in trade, which depends on both the probability of trade per-period and the time between two consecutive periods. We know from proposition 4 that the trading probability per-period converges to zero as the time between two consecutive periods shrinks to zero. Therefore, the expected duration before trade hinges on which of the two converges to zero at a faster rate. The following lemma shows that the search frictions diminish more quickly.

Lemma 1: *When* $z(x, y)$ *is supermodular, as* β *converges to one, the expected time that it takes to trade for any buyer or seller approaches zero.*

To understand why the time to trade converges to zero, let us consider the decision problem of a seller. When the search friction is very small, we know that the seller has already chosen a price such that the matching surplus is close to zero. Suppose the time to trade is still positive and there is a further reduction in the search friction. The seller can benefit from a lower search friction in two ways. He can either keep the same price and reduce the time to trade, or raise the price and expect to trade within the original time frame. Notice that the price increment is close to zero while the reduction in time to trade is strictly positive. Therefore, the seller will respond by reducing the time to trade. In other words, search frictions must diminish at a faster speed than the per-period trading probabilities do. The result also applies to buyers due to the nature of bilateral trade.

In addition, the convergence requires that the limiting matching set becomes efficient. In a perfectly competitive market with no information friction, a buyer who is *a* percentile of the buyer-entrants type distribution only trades with sellers whose type is within the same percentile of the seller-entrants type distribution. The following lemma shows that the limiting matching set has the same property.

Lemma 2: *When $z(x, y)$ is supermodular, as β converges to one, $\Gamma_B(x) - \Gamma_S(m(x))$ converges to zero.*

This lemma is obtained based on two observations. First, even after taking into account the change in the market size as search frictions vanish, the measure of forced exit per-period still shrinks to zero. Second, the sorting converges to one-to-one positive assortative. Then the total measure of buyer exits with types below x is arbitrarily close to the total measure of seller exits with types below $m(x)$. We can then use the steady state condition to obtain the result in the lemma.

We are now ready to establish the convergence to the Walrasian Equilibrium.

Proposition 6: *When $z(x, y)$ is supermodular, as β converges to one, any search equilibrium converges to perfect competition.*

To further understand this convergence result, we utilize the insights from Gretsky, Ostroy, and Zame (1999), i.e. interpreting "perfect competition" as *the inability of individuals to (favorably) influence prices.* More precisely, this will entail sellers facing perfectly elastic demand and buyers facing perfectly elastic supply. Toward formalizing these concepts, let us introduce the notion of a "price elasticity of demand" faced by a type y seller setting price p:

$$E_y(p) = \frac{\left(\frac{\partial(\mu(\mathbb{M}_S(y,p)))}{\mu(\mathbb{M}_S(y,p))}\right)}{\left(\frac{\partial p}{p}\right)}$$

where $\mathbb{M}_S(y, p)$ is the matching set of a seller with type y when she chooses price p. Naturally, this reflects the responsiveness of a seller's trading probability (per period) to her chosen price.

Accordingly, the term "value elasticity of supply" will pertain to the corresponding notion for a type x buyer with continuation value v:

$$E_x(v) = \frac{\left(\frac{\partial(\mu(\mathbb{M}_B(x,v)))}{\mu(\mathbb{M}_B(x,v))}\right)}{\left(\frac{\partial v}{v}\right)}$$

where $\mathbb{M}_B(x, v)$ is the matching set of a buyer with type x when his continuation value is v. In this case, let us imagine that each buyer is choosing an optimal matching set, which is determined implicitly by her choice of a continuation value (taking prices as given). Following this logic, the above object reflects how a buyer's per-period trading probability responds to this value.

For agents to be unable to influence prices, the two elasticities above should be arbitrarily large. The following lemma shows that this is indeed the case.

Lemma 3: *When $z(x, y)$ is supermodular, the price elasticity of demand faced by any seller $E_y(p)$ and the value elasticity of supply faced by any buyer $E_x(v)$ are perfectly elastic.*

5.3 Existence

We have not proved the existence of a search equilibrium when β converges to 1. In this section, we will fill in this gap. The previous section showed that the equilibrium price converges to a function that is continuous in y and independent of $V(x)$. For this reason, we prove the existence of search equilibrium assuming that the pricing strategy is continuous in y and $V(x)$.[9]

We build on the existence proof in Shimer and Smith (2000). Our setting is different from theirs. They assume perfect information, Nash bargaining and symmetry between the two sides and we assume private information on buyer side and full bargaining power on seller side. Despite this, we can still borrow majority of the proof with some adjustments once we assume that the pricing strategy is continuous in y and $V(x)$. The symmetry is not crucial in Shimer and Smith's proof but the continuity of the surplus function is. A continuous pricing strategy is then necessary for the surplus function to be continuous in our setting. The detailed proof is presented in the appendix. In short, what we show in the proof is that the mapping determined by the equilibrium conditions from the continuation payoff $V(x)$ to itself is well defined and continuous. The existence then follows from the Schauder fixed point theorem.

Lemma 4: *When $z(x, y)$ is supermodular, if the pricing strategy is continuous in sellers' types and buyers' value function, then search equilibrium exists.*

The lemma then directly implies the existence of equilibrium in the limit.

[9]The proof itself assumes an arbitrary β. Therefore, it also proves the existence of search equilibrium with any smaller β if one can further verify that the pricing strategy is indeed continuous in equilibrium. In this paper, we restrict attention to the two extreme cases; the characterization of equilibrium with interior search frictions is beyond the scope of this paper. But this proof with an arbitrary β may be useful for future works.

Proposition 7: *When $z(x, y)$ is supermodular, a search equilibrium exists when β is arbitrarily close to* 1*.*

7. Conclusion

The presence of buyer private information does impede sorting when search is frictional, and we have highlighted the relationship between the strength of this effect and the degree of competition in the market. At one extreme, when there is bilateral monopoly power in each buyer-seller meeting, PAM requires the gain from trade $z(x, y) - c(y)$ to be log-supermodular. At the other extreme, when the search frictions vanish, a supermodular match value is sufficient to give rise to PAM. The difference arises because, with private information, sellers face a trade-off between price and trading probability per period when search frictions are positive. One nuance we found is that PAM in a frictional environment may require less complementarity than supermodularity. When the gain $z(x, y) - c(y)$ is decreasing in y, the supermodularity of $z(x, y)$ is sufficient but not necessary for log-supermodularity.

The gain from higher trading probabilities grow inconsequential as the environment approaches the frictionless limit. Thus, as search frictions vanish, the sorting consequences of private information do as well, and the standard supermodularity condition is sufficient to generate positive sorting. Moreover, given the sorting pattern, the increasing competition results in a surplus division that approaches its competitive counterpart: agents from both sides of the market obtain their marginal contributions. The sorting pattern in the limit itself also coincides with that of perfect competition and no agents expect delay in trade in the limit. We show that with information friction and the bargaining protocol that sellers make take-it-or-leave-it offers, the search equilibria converge to perfect competition.

This paper restricts attention to a specific information structure and bargaining protocol. As shown in the earlier example, if buyers have no private information, then search equilibrium does not converge to the competitive limit. For future works, we can consider more general settings and find out the necessary and sufficient conditions for convergence.

Appendix

Proof of Proposition 4
Proof:

Step 1: $d(x,y)=1 \iff s(x,y)\in[0,\xi)$ *for any* $\xi>0$.

The direction "$\Leftarrow$" follows from the construction of function $d(x,y)$.

To see the other direction, notice that $d(x,y)=1$ implies $s(x,y)\geq 0$. Suppose that there exists a $\tilde{\xi}>0$, an $\tilde{x}\in[\underline{x},\bar{x}]$ and a $\tilde{y}\in[\underline{y},\bar{y}]$ such that $s(\tilde{x},\tilde{y})=z(\tilde{x},\tilde{y})-P(\tilde{y})-\beta V(\tilde{x})>\tilde{\xi}$ for any $\beta\in[0,1)$. Since the function $s(x,y)$ is Lipschitz continuous, $s(\tilde{x},\tilde{y})>\tilde{\xi}$ implies that $\mathbb{M}_S(\tilde{y})$ is non-empty. We can then partition $\mathbb{M}_S(\tilde{y})$ into two sets, $\mathbb{M}_S^1(\tilde{y})=\{x: s(x,\tilde{y})\geq\tilde{\xi}_1\}$ and $\mathbb{M}_S^2(\tilde{y})=\{x: s(x,\tilde{y})\in[0,\tilde{\xi}_1]\}$ with some $\tilde{\xi}_1<\tilde{\xi}$, such that the two sets have the same probability measure, which is denoted as Q ($Q\in(0,\frac{1}{2})$).

Seller $\tilde{y}$'s expected profit can then be written as

$$\Pi(\tilde{y})=\frac{2Q(P(\tilde{y})-c(\tilde{y}))}{1-\beta+2\beta Q}=\frac{P(\tilde{y})-c(\tilde{y})}{1+\frac{1-2Q}{2Q}(1-\beta)}$$

If the seller raises the price by $\tilde{\xi}_1$, then the new matching set becomes $\mathbb{M}_S^1(\tilde{y})$. It is profitable to raise the price if the change in profit is positive, i.e.,

$$\tilde{\xi}_1[1+\frac{1-2Q}{2Q}(1-\beta)]-\frac{1}{2Q}(1-\beta)(P(\tilde{y})-c(\tilde{y}))>0$$

This inequality holds in the limit if $\frac{1-\beta}{Q}$ converges to zero. This needs to be verified because the distribution of buyers is endogenously determined. Denote the market size as M and the average trading probability of buyers as $\bar{\mu}(\mathbb{M}_B)$, both of which depend on β. The steady state condition $t=M[1-\beta+\beta\bar{\mu}(\mathbb{M}_B)]$ implies that $\frac{M[1-\beta+\beta\mu(\mathbb{M}_B(x))]}{t}$ is finite or zero for a.e. x. In addition, we know

$$\frac{f_B(x)}{1-\beta}=\frac{\gamma_B(x)}{1-\beta}\frac{t}{M[1-\beta+\beta\mu(\mathbb{M}_B(x))]}$$

The right-hand side of the above equation diverges to infinity a.e.. Therefore, $\frac{f_B(x)}{1-\beta}$ also diverges to infinity a.e.. As a result, $\frac{1-\beta}{Q}$ converges to zero. We have proved the claim.

Step 2: $\mu(\mathbb{M}_B(x))\in[0,\xi)$ *and* $\mu(\mathbb{M}_S(y))\in[0,\xi)$.

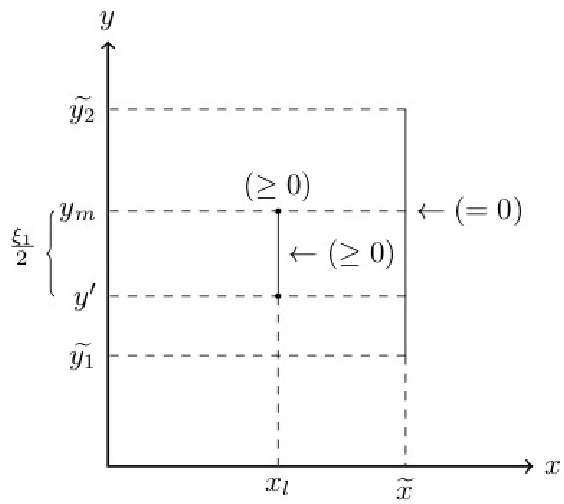

Figure 1: Graphical Illustration for Step 2.2

We discuss the following two kinds of matching sets: 1) the matching sets in which the surplus function is not always zero and 2) the matching sets in which the surplus function is always zero. We will show that the statement holds for both cases.

Step 2.1: consider the matching sets in which $s(x, y)$ is not always zero.

Suppose there exist a $\tilde{\xi} > 0$ and a $\tilde{y} \in [\underline{y}, \bar{y}]$ such that for any $\epsilon \in (0, 1)$, $\mu(\mathbb{M}_S(\tilde{y})) > \tilde{\xi}$ for some $\beta \in [1 - \epsilon, 1)$. Since $\mu(\mathbb{M}_S(\tilde{y})) > \tilde{\xi}$ and the surplus function $s(x, \tilde{y})$ is continuous and not always zero, $s(x, \tilde{y})$ must be bounded above zero for some $x \in \mathbb{M}_S(\tilde{y})$. This contradicts the result in step 1.

Following the same argument, $\mu(\mathbb{M}_B(x)) \in [0, \xi)$ for any $\xi > 0$.

Step 2.2: consider the matching sets in which $s(x, y)$ is zero.

Suppose that there exists an $\tilde{x}$ and a $\xi_1 > 0$, such that for any $\epsilon \in (0, 1)$, the matching set $\mathbb{M}_B(\tilde{x})$ has a subset $[\tilde{y}_1, \tilde{y}_2]$ with $\tilde{y}_2 - \tilde{y}_1 \geq \xi_1$ for some $\beta \in [1 - \epsilon, 1)$, and $s(\tilde{x}, y) = 0$ for any $y \in [\tilde{y}_1, \tilde{y}_2]$. One may refer to Figure 1 for graphical illustration, which shows the values of $s(x, y)$ for particular pairs of agents in the argument to be followed.

Denote y_m as the middle point of the interval, $y_m = \frac{\tilde{y}_1 + \tilde{y}_2}{2}$. We first show that there exist an $\hat{\epsilon} \in (0, 1)$ such that for any $\beta \in [1 - \hat{\epsilon}, 1)$, $\tilde{x}$ is the unique element of $\mathbb{M}_S(y_m)$. Consider any $x_l < \tilde{x}$. Suppose $x_l \in \mathbb{M}_S(y_m)$, i.e., $s(x_l, y_m) \geq 0$. Define $y' = y_m - \frac{\xi_1}{2}$. y' is in the interval $[\tilde{y}_1, \tilde{y}_2]$ because the length of the interval is greater than ξ_1. Therefore

$s(\tilde{x}, y') = 0$. By supermodularity, any $y \in [y', y_m]$ must be an element in $\mathbb{M}_B(x_l)$. Then $\mu(\mathbb{M}_B(x_l)) \geq \mu([y', y_m]) >> 0$.

Next, we show that the surplus function is rarely constant in one variable. Define $N_s(x) = \{y : s(x, y) = 0\}$, $N_s(y) = \{x : s(x, y) = 0\}$ and $N_s = \{(x, y) : s(x, y) = 0\}$. Pick $x \neq x'$ and $y \neq y'$, such that $s(x, y) = s(x', y) = s(x, y') = 0$. If $z(x, y)$ is supermodular, it must be true that $s(x', y') \neq 0$. To see this, notice

$$s(x, y) - s(x', y) = z(x, y) - z(x', y) - \beta(V(x) - V(x'))$$
$$s(x, y') - s(x', y') = z(x, y') - z(x', y') - \beta(V(x) - V(x'))$$

Then $z(x, y) - z(x', y) \neq z(x, y') - z(x', y')$ implies $s(x', y') \neq 0$. Then by the proof in Appendix B of Shimer and Smith (2000), $\mu(N_s(x)) = 0$ for a.e. x, $\mu(N_s(y)) = 0$ for a.e. y and $\mu(N_s) = 0$ a.e..

Because the surplus function is rarely constant in one variable, the result in step 2.1 applies to $\mathbb{M}_B(x_l)$. This leads to a contradiction.

Following the same argument, for large enough β's, any $x_h > \tilde{x}$ is not in $\mathbb{M}_S(y_m)$.

Therefore, for any large enough β, $\mathbb{M}_S(y_m)$ contains only one point $\tilde{x}$. Since the probability of meeting a buyer of type $\tilde{x}$ is zero, the profit of the seller of type y_m is zero. This is a contradiction.

The same proof can be used to show that $\mu(\mathbb{M}_S(y)) \in [0, \xi)$ for any $\xi > 0$.

Step 3: converge to one-to-one positive sorting.

Step 3.1: all agents have positive trading probability.

It is easy to see that buyers' equilibrium payoff is an increasing function. This means that the matching set of any buyer has strictly positive measure if there exists at least one lower type whose trading probability is positive. Denote x_L as the highest type whose matching set has measure 0.

We have shown in step 1 that $\frac{f_B(x)}{1-\beta}$ diverges to ∞ for a.e. x. Therefore, $\frac{\bar{\mu}(\mathbb{M}_S)}{1-\beta}$ diverges to ∞, where $\bar{\mu}(\mathbb{M}_S)$ denotes the average trading probability of sellers. Then the steady state condition $t = M[1 - \beta + \beta\bar{\mu}(\mathbb{M}_S)] = M(1-\beta)[1 + \beta\frac{\bar{\mu}(\mathbb{M}_S)}{1-\beta}]$ implies that $\frac{M(1-\beta)}{t}$ converges to 0. Because the matching sets of buyers of type $x < x_L$ has measure 0, we have

$$F_B(x_L)M(1-\beta) = t\Gamma_B(x_L)$$

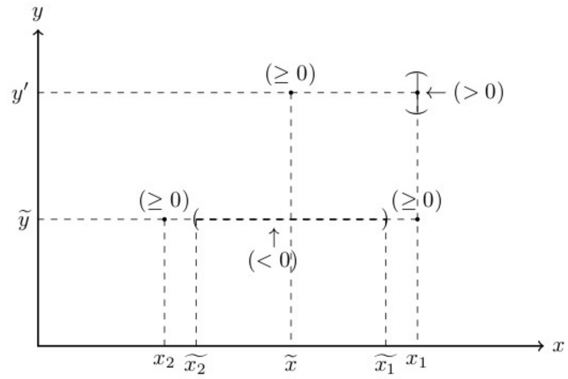

Figure 2: Graphical Illustration for Step 3.2

If x_L is strictly larger than $\underline{x}$ so that $\Gamma_B(x_L)$ is strictly positive, $F_B(x_L)$ goes to infinity in the limit. This contradicts the fact that $F_B(x_L) \in [0, 1]$.

Next, it is easy to show that all sellers have matching sets with positive measures. The reason is that, compared to no trade, any seller would strictly prefer to set a price slightly above $c(y)$ so that given the assumption of $z(x, y) - c(y) > 0$ for any x, the lowest type buyers with $V(x)$ close to zero will accept the deal.

Step 3.2: the distance between any two elements in a matching set is arbitrarily small.

Suppose that there exists a $\tilde{y}$ and an $\eta > 0$, such that for any $\epsilon > 0$, there is a $\beta \in [1-\epsilon, 1)$. With this β, we can find $x_1, x_2 \in \mathbb{M}_S(\tilde{y})$ with $x_1 - x_2 \geq \eta$. One may refer to Figure 2 for graphical illustration, which shows the values of $s(x, y)$ for given pairs of agents in the argument to be followed. Then we have $x_1 > \tilde{x}_1 > \tilde{x}_2 > x_2$ such that $s(x, \tilde{y})$ is bounded below zero for any $x \in (\tilde{x}_2, \tilde{x}_1)$. Then for any seller of type $y' > \tilde{y}$, if her matching set contains some $\tilde{x} \in (\tilde{x}_2, \tilde{x}_1)$, supermodularity implies that a neighborhood of y' must be a subset of $\mathbb{M}_B(x_1)$. This means that the measure of $\mathbb{M}_B(x_1)$ must be bounded above zero, which contradicts the result in step 2. The same argument also applies to any seller of type $y' < \tilde{y}$. Therefore, $\mathbb{M}_B(x) = \emptyset$ for any $x \in (\tilde{x}_2, \tilde{x}_1)$. However, this contradicts the result in Step 3.2. Therefore, for any y and $\eta > 0$, there exists an $\epsilon > 0$, such that for any $\beta > 1 - \epsilon$, it must be true that $x_1 - x_2 < \eta$ for any $x_1 > x_2 \in \mathbb{M}_S(y)$. The same argument applies to $\mathbb{M}_B(x)$.

Step 3.3: the existence of $m(x)$ function.

Pick any x, y, y' such that $d(x, y) = 1$ and $y' > y$. We show that under the assumption of supermodularity, there must exist an $x' \in \mathbb{M}_S(y')$ such that for any $\epsilon > 0$, $x' \geq x - \epsilon$.

Suppose otherwise. Define $\tilde{x}'$ as $\sup\{\mathbb{M}_S(y')\}$. Then there exists a $\delta > 0$, such that $x - \tilde{x}' > \delta$. By supermodularity, $s(\tilde{x}', y) > 0$, which implies that $\mathbb{M}_S(y)$ has two elements with distance greater than δ. This contradicts the result in Step 3.2. ■

Proof of Proposition 5

Proof:

To prove the proposition, we first show that the following lemma is true.

Lemma 5: *When $z(x, y)$ is supermodular, $V(x)$ is non-negative, increasing in x and Lipschitz continuous in equilibrium. When $\beta \to 1$, $V(x)$ is differentiable and the derivative satisfies*

$$V'(x) \to z_1(x, m(x)).$$

Proof: $V(x)$ can be rearranged and expressed as

$$V(x) = \frac{1}{1-\beta} \int_{\mathbb{M}_B(x)} (z(x, y) - P(y) - \beta V(x)) f_S(y) dy$$

$V(x)$ is non-negative because any $y \in \mathbb{M}_B(x)$ satisfies $s(x, y) \geq 0$. Also, any $M \neq \mathbb{M}_B(x)$ either excludes $y \in \mathbb{M}_B(x)$, in which case $z(x, y) - P(y) - \beta V(x) > 0$, or includes $y \notin \mathbb{M}_B(x)$, in which case $z(x, y) - P(y) - \beta V(x) < 0$. Therefore, for any M,

$$V(x) \geq \frac{1}{1-\beta} \int_{M} (z(x, y) - P(y) - \beta V(x)) f_S(y) dy$$

Consider any $x_2 \geq x_1$,

$$\begin{aligned} &(1-\beta)[V(x_2) - V(x_1)] \\ = &\int_{\mathbb{M}_B(x_2)} (z(x_2, y) - P(y) - \beta V(x_2)) f_S(y) dy - \int_{\mathbb{M}_B(x_1)} (z(x_1, y) - P(y) - \beta V(x_1)) f_S(y) dy \\ \geq &\int_{\mathbb{M}_B(x_1)} [z(x_2, y) - z(x_1, y) - \beta(V(x_2) - V(x_1))] f_S(y) dy \end{aligned}$$

This implies

$$V(x_2) - V(x_1) \geq \frac{\int_{\mathbb{M}_B(x_1)} [z(x_2, y) - z(x_1, y)] f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}_B(x_1)} f_S(y) dy} \geq 0$$

That is, $V(x)$ increases in x.

Similarly, we can obtain

$$V(x_2) - V(x_1) \leq \frac{\int_{\mathbb{M}_B(x_2)}[z(x_2, y) - z(x_1, y)]f_S(y)dy}{1 - \beta + \beta \int_{\mathbb{M}_B(x_2)} f_S(y)dy}$$

By the assumption that $z(x, y)$ is Lipschitz continuous, there exists a real constant κ such that $\mid z(x_2, y) - z(x_1, y) \mid \leq \kappa(x_2 - x_1)$. Combined with the above two inequalities

$$\frac{-\kappa(x_2 - x_1) \int_{\mathbb{M}_B(x_1)} f_S(y)dy}{1 - \beta + \beta \int_{\mathbb{M}_B(x_1)} f_S(y)dy} \leq V(x_2) - V(x_1) \leq \frac{\kappa(x_2 - x_1) \int_{\mathbb{M}_B(x_2)} f_S(y)dy}{1 - \beta + \beta \int_{\mathbb{M}_B(x_2)} f_S(y)dy}.$$

Therefore, $\mid V(x_2) - V(x_1) \mid \leq \kappa(x_2 - x_1)$, which implies that $V(x)$ is Lipschitz continuous.

Consider $x' \neq x$.

$$\begin{aligned}(1-\beta)\frac{V(x') - V(x)}{x' - x} &= \int_{\mathbb{M}_B(x') - \mathbb{M}_B(x)} \frac{z(x', y) - P(y) - \beta V(x')}{x' - x} f_S(y)dy \\ &+ \int_{\mathbb{M}_B(x)} [\frac{z(x', y) - z(x, y)}{x' - x} - \beta \frac{V(x') - V(x)}{x' - x}] f_S(y)dy\end{aligned}$$

Take the limit as $x' \to x$. The first integral on the right-hand side vanishes because 1) $\mathbb{M}_B(x') - \mathbb{M}_B(x)$ is arbitrarily small as $\beta \to 1$, and 2) $\lim \frac{s(x',y)}{x'-x} < \lim \frac{s(x',y) - s(x,y)}{x'-x} = \lim \frac{z(x',y) - z(x,y) - \beta V(x') + \beta V(x)}{x'-x} < \infty$, where the first inequality follows because $s(x, y) < 0$. Rearranging terms, we get

$$V'(x) = z_1(x, \tilde{y}) \frac{\mu(\mathbb{M}_B(x))}{1 - \beta + \beta\mu(\mathbb{M}_B(x))} \text{ for some } \tilde{y} \in \mathbb{M}_B(x)$$

Following the same argument as in the proof for Proposition 4, the steady state condition $t = M[1 - \beta + \beta\bar{\mu}(\mathbb{M}_S)]$ implies that $\frac{f_S(y)}{1-\beta}$ diverges to ∞ for a.e. y. We have also shown that the matching set of any buyer is non-empty in the limit. Therefore, $\frac{1-\beta}{\mu(\mathbb{M}_B(x))} \to 0$ as $\beta \to 1$ for any x. Therefore,

$$V'(x) \to z_1(x, m(x)).$$

■

The lemma implies

$$\begin{aligned}V(x) \to & V(\underline{x}) + \int_{\underline{x}}^{x} z_1(\tilde{x}, m(\tilde{x}))d\tilde{x} \\ = & V(\underline{x}) + \int_{\underline{x}}^{x} dz(\tilde{x}, m(\tilde{x})) - \int_{\underline{y}}^{m(x)} z_2(m^{-1}(\tilde{y}), \tilde{y})d\tilde{y}\end{aligned}$$

Since $s(x, m(x)) \to 0$, for any $y \in [\underline{y}, \bar{y}]$, the price charged by a seller of type $y = m(x)$ has the following limit:

$$\begin{aligned}P(y) \to & z(m^{-1}(y), y) - V(m^{-1}(y)) \\ \to & z(\underline{x}, \underline{y}) + \int_{\underline{x}}^{x} dz(\tilde{x}, m(\tilde{x})) - [(V(\underline{x}) + \int_{\underline{x}}^{x} dz(\tilde{x}, m(\tilde{x})) - \int_{\underline{y}}^{m(x)} z_2(m^{-1}(\tilde{y}), \tilde{y})d\tilde{y}] \\ = & z(\underline{x}, \underline{y}) - V(\underline{x}) + \int_{\underline{y}}^{y} z_2(m^{-1}(\tilde{y}), \tilde{y})d\tilde{y}\end{aligned}$$

Here, $V(\underline{x}) = 0$ because sellers have all the bargaining power. ■

Proof of Lemma 1

Proof: The expected time to trade for a buyer of type x (seller of type y) is proportional to $\frac{1-\beta}{\mu(\mathbb{M}_B(x))}$ ($\frac{1-\beta}{\mu(\mathbb{M}_S(y))}$). We know from the proof of Proposition 5 that $\frac{1-\beta}{\mu(\mathbb{M}_B(x))}$ converges to 0 for any x. By the same argument, $\frac{1-\beta}{\mu(\mathbb{M}_S(y))} \to 0$ follows from the fact that $\frac{f_B(x)}{1-\beta}$ diverges to ∞ for a.e. x. ■

Proof of Lemma 2

Proof: From the steady-state distribution, we know that for any $x > x_L$,

$$\Gamma_B(x) = \Gamma_B(x_L) + \int_{x_L}^{x} [1 - \beta + \beta\mu(\mathbb{M}_B(x))] \frac{M}{t} f_B(x)dx$$

Because $\frac{M}{t}(1-\beta)$ converges to zero, $[1 - \beta + \beta\mu(\mathbb{M}_B(x))] \frac{M}{t}$ converges to $\mu(\mathbb{M}_B(x))\frac{M}{t}$. In addition, we know that $x_L \to \underline{x}$. Therefore,

$$\Gamma_B(x) \to \int_{\underline{x}}^{x} \mu(\mathbb{M}_B(x))\frac{M}{t} f_B(x)dx$$

Similarly,

$$\Gamma_S(y) \to \int_{\underline{y}}^{y} \mu(\mathbb{M}_S(y))\frac{M}{t} f_S(y)dy$$

Note that the right-hand sides of the above two expressions are the total measure of trade for buyers with types lower than x and for sellers with types lower than y within one unit of time. As β approaches 1, these two measures of trades must be arbitrarily close. This implies

$$\Gamma_B(x) - \Gamma_S(m(x)) \to 0$$

■

Proof of Lemma 3

Proof:

A seller has no incentive to change the price if

$$\frac{d\mu(\mathbb{M}_S(y))}{\mu(\mathbb{M}_S(y))} \leq -\frac{1-\beta+\beta\mu(\mathbb{M}_S(y))}{1-\beta}\frac{P(y)}{P(y)-c(y)}\frac{-dP(y)}{P(y)}$$

Using the previous results, it is easy to show that $\frac{1-\beta+\beta\mu(\mathbb{M}_S(y))}{1-\beta}$ converges to ∞. As a result, $E_y(p)$ converges to $-\infty$.

Following the same approach, we can show that a buyer has no incentive to change its matching set by varying $V(x)$ if

$$E_x(v) \leq -\frac{1-\beta+\beta\mu(\mathbb{M}_B(x))}{1-\beta}$$

The right-hand-side again converges to $-\infty$ as β converges to one. ■

Proof of Lemma 4

We first prove Lemma 6 and 7.

Lemma 6: *When $z(x,y)$ is supermodular, for any price function $P(y,V(x))$ that is continuous in y and $V(x)$, any Borel measurable mapping from the buyer's value functions $V(x)$ to the match indicator function $d(x,y)$ is continuous.*

Proof: Define $N_s = \{(x,y) : s(x,y) = 0\}$ and $\sum_s(\eta) = \{(x,y) : |s(x,y)| \in [0,\eta]\}$. The latter set shrinks monotonically to $\cap_{k=1}^{\infty}\sum_s(1/k) = N_s$.

$$\lim_{\eta\to 0}(\mu\times\mu)(\sum_s(\eta)) = (\mu\times\mu)(\cap_{k=1}^{\infty}\sum_s(1/k)) = (\mu\times\mu)(N_s) = 0$$

The last equality follows from the result proved earlier that $s(x,y)$ is rarely constant in one variable. Let V^1 and V^2 be two value functions, and d^1 and d^2 be the corresponding match indicator functions.

Since $P(y,V)$ is continuous in V, for any $\epsilon > 0$, there exists a $\eta' > 0$, such that

$$\beta \parallel V^1(x) - V^2(x) \parallel < \eta' \Rightarrow \mid P(y,V^1) - P(y,V^2) \mid < \epsilon, \text{ for any } y$$

In other words, we can always pick close enough value functions such that the price functions are close. Let $\eta = 2\max\{\eta', \epsilon\}$. If $s^1(x,y) = z(x,y) - \beta V^1(x) - P(y,V^1) > \eta$, then $s^2(x,y) = z(x,y) - \beta V^2(x) - P(y,V^2) > 0$. So $d^1(x,y) = d^2(x,y) = 1$. By the same logic, If $s^1(x,y) < -\eta$, then $s^2(x,y) < 0$. So $d^1(x,y) = d^2(x,y) = 0$. As a result, $\{(x,y) : d^1(x,y) \neq d^2(x,y)\} \subseteq \sum_{s^1}(\eta)$. The Lebesgue measure of $\sum_{s^1}(\eta)$ vanishes as $\eta \to 0$. The continuity is thus established

$$\lim_{\|V^1(x)-V^2(x)\| \to 0} \parallel d^1(x,y) - d^2(x,y) \parallel_{\mathcal{L}^1} = 0.$$

■

Lemma 7: *The mapping $d(x,y) \to (f_B(x), f_S(y))$ is well defined and continuous.*

Proof:

Step 1: The mapping is well defined.

Given entrant densities $\gamma_B(x)$ and $\gamma_S(y)$, the mapping is well defined if there exist unique functions $\hat{f}_B$ and $\hat{f}_S$ that solve the following system of equations,

$$\hat{f}_B(x) = \frac{t\gamma_B(x)}{1 - \beta + \beta\frac{\int d(x,y)\hat{f}_S(y)dy}{\int \hat{f}_S(y)dy}}$$
$$\hat{f}_S(y) = \frac{t\gamma_S(y)}{1 - \beta + \beta\frac{\int d(x,y)\hat{f}_B(x)dx}{\int \hat{f}_B(x)dx}}$$

It is easy to see that $\hat{f}_B(x) \in [t\gamma_B(x), t\gamma_B(x)/(1-\beta)]$ and $\hat{f}_S(y) \in [t\gamma_S(y), t\gamma_S(y)/(1-\beta)]$, where the two upper bounds are finite as $\gamma_B(x)$ and $\gamma_S(y)$ are finite and $\frac{t}{1-\beta} \to \frac{1}{r}$ as $t \to 0$.

One can apply a log transformation method similar to that used in Shimer and Smith

(2000) and reformulate the problem as a fixed-point problem.

$$
\begin{aligned}
\Phi_B(h) &= \log \frac{t\gamma_B(x)}{1-\beta+\beta\frac{\int d(x,y)e^{h_S(y)}dy}{\int e^{h_S(y)}dy}} \\
\Phi_S(h) &= \log \frac{t\gamma_S(y)}{1-\beta+\beta\frac{\int d(x,y)e^{h_B(x)}dx}{\int e^{h_B(x)}dx}}
\end{aligned}
$$

where $h_B(x) = \log \hat{f}_B(x)$, $h_S(y) = \log \hat{f}_S(y)$, $h = (h_B, h_S)'$. The mapping is well defined if $\Phi(h) = h$ has a unique fixed point. We prove this using the Contraction Mapping Theorem. Consider h^1 and h^2,

$$
\begin{aligned}
\Phi_B(h^2) - \Phi_B(h^1) &= \log \frac{1-\beta+\beta\frac{\int d(x,y)e^{h_S^1(y)}dy}{\int e^{h_S^1(y)}dy}}{1-\beta+\beta\frac{\int d(x,y)e^{h_S^2(y)}dy}{\int e^{h_S^2(y)}dy}} \\
&\le \log \frac{1-\beta+\beta e^{\|h_S^1-h_S^2\|}\frac{\int d(x,y)e^{h_S^2(y)}dy}{\int e^{h_S^2(y)}dy}}{1-\beta+\beta\frac{\int d(x,y)e^{h_S^2(y)}dy}{\int e^{h_S^2(y)}dy}} \\
&\le \log \frac{1-\beta+\beta e^{\|h_S^1-h_S^2\|}}{1-\beta+\beta} \\
&= \log[1-\beta+\beta e^{\|h_S^1-h_S^2\|}]
\end{aligned}
$$

The first inequality follows because $e^{\|h_S^1-h_S^2\|} > e^{h_S^1(y)-h_S^2(y)}$ for any y while the second inequality holds because $e^{\|h_S^1-h_S^2\|} > 1$ and hence the term is increasing in $\frac{\int d(x,y)e^{h_S^2(y)}dy}{\int e^{h_S^2(y)}dy}$. We thus have

$$
\frac{\Phi_B(h^2) - \Phi_B(h^1)}{\| h_S^1 - h_S^2 \|} \le \frac{\log[1-\beta+\beta e^{\|h_S^1-h_S^2\|}]}{\| h_S^1 - h_S^2 \|}
$$

In addition, we know that $h_S(y) \in [\log(t\gamma_S(y)), \log(t\gamma_S(y)) - \log(1-\beta)]$, which implies $\| h_S^1 - h_S^2 \| \in [0, -\log(1-\beta)]$. Since the right hand side of the above inequality increases in $\| h_S^1 - h_S^2 \|$,

$$
\frac{\Phi_B(h^2) - \Phi_B(h^1)}{\| h_S^1 - h_S^2 \|} \le \frac{\log[1-\beta+\frac{\beta}{1-\beta}]}{\log\frac{1}{1-\beta}} = \chi \in (0,1)
$$

The same argument applies in the other direction and we thus obtain

$$\frac{\| \Phi_B(h^1) - \Phi_B(h^2) \|}{\| h_S^1 - h_S^2 \|} \leq \chi$$

We have the symmetric inequality for y. Denote $\Phi(h) = (\Phi_B(h)\Phi_S(h))'$. Combining the two inequalities,

$$\| \Phi(h^1) - \Phi(h^2) \| \leq A \| h^1 - h^2 \|$$

where A is a matrix with $| A |= -\chi^2 \in (-1, 1)$. We have thus proven that it is a contraction mapping.

Step 2: The mapping is continuous.

Define $G_B(d, \hat{f})(x) = \hat{f}_B(x)[1 - \beta + \beta \int d(x, y)\frac{\hat{f}_S(y)}{\int \hat{f}_S(y)dy}dy] - t\gamma_B(x)$ and $G_S(d, \hat{f})(y) = \hat{f}_S(y)[1 - \beta + \beta \int d(x, y)\frac{\hat{f}_B(x)}{\int \hat{f}_B(x)dx}dx] - t\gamma_S(y)$, $G(d, \hat{f}) = (G_B(d, \hat{f}), G_S(d, \hat{f}))$. In equilibrium, $G(d, \hat{f}) = 0$

Suppose that there exist d^1 and d^2 with $\| d^1 - d^2 \|_{\mathcal{L}^1} \to 0$, such that $\| \hat{f}^1 - \hat{f}^2 \|_{\mathcal{L}^1} \nrightarrow 0$. Then there exists $\hat{\epsilon} > 0$ such that $\| G(d^1, \hat{f}^2) \|_{\mathcal{L}^1} > \hat{\epsilon}$. WOLG, assume $\| G_B(d^1, \hat{f}^2) \|_{\mathcal{L}^1} > \epsilon$.

On the other hand,

$$\begin{aligned}
\| G_B(d^1, \hat{f}^2) \|_{\mathcal{L}^1} &= \| G_B(d^1, \hat{f}^2) - G_B(d^2, \hat{f}^2) \|_{\mathcal{L}^1} \\
&= \| \hat{f}_B^2(x)\beta \int (d^1(x, y) - d^2(x, y))\frac{\hat{f}_S^2(y)}{\int \hat{f}_S^2(y)dy}dy \|_{\mathcal{L}^1} < \epsilon
\end{aligned}$$

The last line follows since $\frac{\hat{f}_S^2(y)}{\int \hat{f}_S^2(y)dy}$ and $\hat{f}_B^2(x)$ are bounded for any x and y. This leads to a contradiction.

■

We are now ready to prove the existence of the equilibrium.

Proof:

Equilibrium exists if $T(V) = V$ has a unique fixed point, where,

$$T(V) = \int \max\{z(x, y) - P(y, V), \beta V(x)\}f_S^V(y)dy$$

Here, $f_S^V(y)$ is the density function of sellers when buyers have value function V. Following the Schauder Fixed Point Theorem, we need a nonempty, closed, bounded and convex domain ψ such that,

1. $T : \psi \to \psi$.
2. $T(\psi)$ is an equicontinuous family.
3. T is a continuous operator.

Let ψ be the space of Lipschitz continuous functions V on $[\underline{x}, \bar{x}]$, with lower bound 0 and upper bound $\sup_{x,y} z(x, y)$. Clearly, ψ is nonempty, closed, bounded and convex. Next, we check that the above three requirements hold.

Step 1: $T : \psi \to \psi$ and $T(\psi)$ is an equicontinuous family.

Take any x_1 and x_2 with $x_1 \neq x_2$

$$
\begin{aligned}
& \mid T(V(x_2)) - T(V(x_1)) \mid \\
\leq & \int \mid \max\{z(x_2, y) - P(y, V), \beta V(x_2)\} - \max\{z(x_1, y) - P(y, V), \beta V(x_1)\} \mid f_S^V(y)dy \\
\leq & \int \mid \max\{z(x_2, y) - z(x_1, y), \beta(V(x_2) - V(x_1))\} \mid f_S^V(y)dy
\end{aligned}
$$

Since both $z(x, y)$ and $V(x)$ are Lipschitz-continuous, $T(V)$ is Lipschitz-continuous, which implies equicontinuous. Moreover, it is easy to see that $T(V) \in [0, \sup_{x,y} z(x, y)]$ This also establishes that T is a mapping from ψ to ψ.

Step 2: T is continuous.

Take any V^1 and V^2 with $V^2 \neq V^1$ in ψ. For any x,

$$
\begin{aligned}
& \mid T(V^2(x)) - T(V^1(x)) \mid \\
= & \mid \int \max\{z(x, y) - P(y, V^2), \beta V^2(x)\} f_S^{V^2}(y)dy - \int \max\{z(x, y) - P(y, V^1), \beta V^1(x)\} f_S^{V^1}(y)dy \mid \\
\leq & \mid \int \max\{z(x, y) - P(y, V^2), \beta V^2(x)\} f_S^{V^2}(y)dy - \int \max\{z(x, y) - P(y, V^2), \beta V^2(x)\} f_S^{V^1}(y)dy \mid \\
& + \mid \int (\max\{z(x, y) - P(y, V^2), \beta V^2(x)\} - \max\{z(x, y) - P(y, V^1), \beta V^1(x)\}) f_S^{V^1}(y)dy \mid \\
= & D_1(x) + D_2(x)
\end{aligned}
$$

For $D_1(x)$

$$D_1(x) \leq \int \max\{z(x,y) - P(y,V^2), \beta V^2(x)\} \mid f_S^{V^2}(y) - f_S^{V^1}(y) \mid dy$$
$$\leq \sup_{x,y}[\max\{z(x,y) - P(y,V^2), \beta V^2(x)\}] \int \mid f_S^{V^2}(y) - f_S^{V^1}(y) \mid dy$$

Since $f_S(y)$ is continuous in V, as $\| V^2 - V^1 \| \to 0$, $D_1(x) \to 0$.

For $D_2(x)$

$$D_2(x) \leq \int \mid \max\{z(x,y) - P(y,V^2), \beta V^2(x)\} - \max\{z(x,y) - P(y,V^1), \beta V^1(x)\} \mid f_S^{V^1}(y) dy$$
$$\leq \int \mid \max\{P(y,V^1) - P(y,V^2), \beta V^2(x) - \beta V^1(x)\} \mid f_S^{V^1}(y) dy$$

Since $P(y,V)$ is continuous in V, $D_2(x) \to 0$ as $\| V^2 - V^1 \| \to 0$. ■

Proof of Proposition 7

Proof:

We have already shown that

$$P(y) \to z(\underline{x}, \underline{y}) + \int_{\underline{y}}^{y} z_2(m^{-1}(\tilde{y}), \tilde{y}) d\tilde{y},$$

which is continuous in y. In addition, the function $m(x)$ is solely determined by Γ_B and Γ_S in the limit and is therefore independent of $V(x)$. This implies that the equilibrium price function is continuous in $V(x)$ in the limit. Then by Lemma 4, search equilibrium exists in the limit. ■

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