Long-Term Competition for Product Awareness with Learning from Friends

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Abstract

We study a dynamic model of price competition with differentiated products in which new generations of consumers acquire information about available products from their friends of previous generations. The social network, which links consumers across generations, affects the evolution of consumers' awareness of products and firms' long-term (steady-state) market shares. Focusing on steady-state equilibria, we examine how the structure of the social network - including connectivity and homophily - influences market shares, pricing, and welfare.

Key Words: Learning from friends; Social network; Price competition; Differentiated products; Steady state

JEL Codes: D83; D85; L13; L14

1 Introduction

In markets with differentiated products, consumers are often not fully aware of all available products. One important channel through which consumers learn about available products is by learning from friends who have previously purchased one of the products. This process of learning from friends, or "word of mouth," greatly affects consumers' purchasing behavior as shown by empirical evidence.[1] Given the importance of learning from friends, it is natural to study how the linking pattern among friends, or the structure of social networks, affects competition between firms and the resulting social welfare.

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[1] For empirical evidence regarding the effect of "word of mouth," see Keaveney (1995) on banking, Chevalier and Mayzlin (2006) on book sales, Chintagunta et al. (2010) on entertainment, and Luca (2016) on restaurant choices.

Campbell (2019) provides the first model addressing the above question. Specifically, in his model there are two symmetric firms selling differentiated products and two generations of consumers, with the new generation learning from the old generation about the available products, and the linking pattern between the two generations described by a social network. His model is essentially a static one for the following reasons. First, the purchasing behavior (and information status) of the old generation is exogenously given. Second, firms compete only for the new generation of consumers once.

This paper extends Campbell (2019) into a *dynamic setting*. In particular, there is a sequence of generations of consumers, and the linking pattern between each adjacent generations is described by a social network. The purchasing behavior and information status of each generation are endogenously determined. For each generation of consumers, they make purchasing decisions given their information status; and then their purchasing decisions, through the social network, affect the information status of the next generation, and so forth. The structure of the social network is important in that it influences the joint evolution of consumers' information status and purchasing behavior. Incorporating this dynamic learning process not only makes the model more relevant to real-world situations, but also qualitatively overturns the predictions that are drawn from the static analysis in Campbell (2019).

Our model has two long-lived firms located at the opposite ends on a Hotelling line and competing in prices. We allow the two firms to be asymmetric, say one firm has quality advantage. Each generation of consumers is uniformly distributed on the Hotelling line and lives for one period only. Each consumer has a unit demand, and a necessary condition for a consumer to buy a product is that he is aware of that product. For each generation, a fixed proportion of consumers is exogenously fully informed (aware of both firms' products). For the remaining (endogenous) consumers, they learn about the existence of products from their friends of the previous generation: an endogenous consumer becomes aware of a product if he has a friend of the previous generation who purchased that product.[2] As a result, some endogenous consumers are partially informed (aware of one product only). The number of friends a consumer has is governed by the structure of the social network. In the basic model, we focus on the case of *random connections*, in which the location of each friend is uniformly drawn at random. We then extend the model to the case of *homophily*, under which friends are more likely to have similar locations.

The structure of the network determines how the purchasing behavior of one generation translates into the distribution of information status of the next generation, which in turn determines the purchasing behavior of that generation. We focus on steady-state (long-term) market shares: once reached, they no longer change across generations of consumers. Firms

[2] We assume that each consumer has at least one friend and thus is aware of at least one product.

set prices at the very beginning, which remain fixed in all later periods. Firms' prices induce steady-state market shares through learning from friends, and we assume that firms' objectives are to maximize their steady-state (long-term) profits.

We show that the structure of the network affects firms' pricing decisions in a non-trivial way through the dynamic social learning process. In our model, the firms compete not only for the fully informed consumers in the current period, but also for the partially informed consumers in future periods. By setting a lower price and expanding its full-information market share, through dynamic social learning, a firm gains more partially informed consumers who are aware of its product only in future periods. Interpreting more broadly, in our model current market share serves as a sort of "advertising" through the word-of-mouth learning: a bigger market share today means more consumers will be aware of a firm's product in the future, which implies an even bigger market share in the future. Due to this channel, the intensity of competition crucially depends on how sensitive the steady-state demand is to the full-information market share, which in turn depends on the structure of the social network.

Our formal analysis mainly focuses on the case with symmetric firms. In the basic model with random connections, we derived the following main results. First, under general networks competition is more intense compared to the Hotelling benchmark. Second, the intensity of competition is non-monotonic in network connectivity (the number of friends consumers have). In particular, under the least connected network (the single-friend network under which each consumer has exactly one friend) and the most connected network (the infinite-friend network under which each consumer has an infinite number of friends), the equilibrium price is the highest and coincides with that in the Hotelling benchmark. Under any other generic network, the equilibrium price is lower, and the equilibrium price is non-monotonic in network connectivity.[3] Finally, consumer surplus is also non-monotonic in network connectivity. The above results are qualitatively different from those in Campbell (2019). Specifically, in his model competition is less intense compared to the Hotelling benchmark, and a more connected network always intensifies competition and improves consumer welfare.

The differences in predictions is mainly driven by the dynamic learning channel in our model. With the dynamic learning process, firms also compete for partially informed consumers in the future by affecting the full-information market share, a feature absent in Campbell's (2019) static analysis. This leads to more intense competition in our model than in Campbell's model where firms compete only for fully informed consumers in one period. The dynamic learning process is also responsible for our non-monotonicity result. In particular, an increase in network connectivity induces two effects. First, as the network becomes more connected or each consumer has more friends, more consumers become fully informed and there are

[3] Among the *k-friend* networks (each consumer has exactly k friends), the price is the lowest under the two-friend network and increases in k for $k \geq 2$.

fewer partially informed consumers. Consequently, the competition for the partially informed consumers in future periods is softened. We refer to this as the *volume effect* of network connectivity. On the other hand, network connectivity also has the following *ratio effect*. As the network becomes more connected, for the same increase in a firm's full-information market share (from the equal split in equilibrium), among the partially informed consumers the ratio of the consumers who are informed of the product of the firm in question only to those who are informed of the other firm's product only increases. This ratio effect tends to make the steady-state demand more sensitive to the full-information market share and intensifies competition. These two effects work in opposite directions, giving rise to the non-monotonicity of the intensity of competition in network connectivity.

The non-monotonicity of consumer surplus in network connectivity is mainly due to the non-monotonicity of price. Intuitively, as the network becomes more connected, each consumer gets "more" information, and thus more consumers are fully informed and fewer consumers buy "wrong" (less preferred) products. This information effect improves total welfare and consumer surplus. However, network connectivity also affects the equilibrium price. Since the equilibrium price is non-monotonic in network connectivity, consumer surplus is also non-monotonic. In particular, under sparsely-connected networks, the equilibrium price decreases in connectivity, which benefits consumers; thus consumer surplus is increasing in connectivity. However, under relatively well-connected networks, the equilibrium price increases in connectivity, which hurts consumers. Under such networks, we find that the pricing effect dominates the information effect, so that overall consumer surplus is decreasing in network connectivity.

We also find that an increase in the proportion of exogenously fully informed consumers leads to a higher equilibrium price. The underlying reason is that, with a larger proportion of exogenously fully informed consumers, there are fewer partially informed consumers in the future to compete for, which softens competition. This is a surprising result as it is the opposite to the prediction in standard models (e.g., Varian, 1980), where more fully informed consumers intensifies competition and lowers prices. The reason behind different predictions is again that in standard models firms compete for fully informed consumers only, while in our dynamic model firms also compete for partially informed consumers due to dynamic learning.

In the extended model with homophily, our main finding is that the equilibrium price is monotonically increasing in the degree of homophily; that is, homophily softens competition. As a result, consumer surplus could be decreasing in the degree of homophily. These results are qualitatively from those in Campbell (2019). In particular, his model predicts that the degree of homophily does not affect the equilibrium price and homophily always improves consumer welfare.

Again, the differences in predictions is mainly due to the dynamic learning process in our model. Intuitively, homophily leads to the following information effect: with homophily consumers are more likely to be aware of the "right" (preferred) products, since their friends are more likely to have similar tastes and thus have bought the "right" products; in other words, the "quality" of information received from friends improves. This means that, even if a firm cuts price and expands its full-information market share, it will induce fewer partially informed consumers to "wrongly" buy its product. Therefore, a higher degree of homophily softens the competition for partially informed consumers and raises price.[4] As to consumer welfare, the information effect of homophily always benefits consumers, as homophily improves the quality of information that consumers receive from their friends. However, since the pricing effect works against the information effect, consumer surplus could decrease in the degree of homophily, which is indeed the case when the degree of homophily is not too large.

In the case with asymmetric firms, we find that the dynamic learning process amplifies the advantage of the advantaged firm (firm 1). Specifically, compared to the Hotelling benchmark, in equilibrium firm 1 has a larger market share and firm 2 has a smaller market share and a lower price. The underlying reason is that with a full-information market share bigger than 1/2, firm 1 can gain additional partially informed consumers through the dynamic learning process, which also forces firm 2 to reduce its price. We also find that homophily dampens the advantage of the advantaged firm: firm 1's equilibrium market share is decreasing in the degree of homophily. The underlying reason is again that homophily dampens the dynamic learning effect among partially informed consumers.

Related Literature As mentioned earlier, the closest paper to ours is Campbell (2019). By making the model dynamic, our paper generates predictions that are qualitatively different from those in Campbell (2019). Another difference is that Campbell (2019) focuses solely on symmetric firms, while our paper also considers the case of asymmetric firms.

More broadly, our paper is related to several strands of literature in industrial organization that study settings in which some consumers are not fully informed about available products or prices. One strand of literature studies consumers' search for product information (e.g., Varian, 1980; Wolinsky, 1986; Stahl, 1989). Another strand considers firms' advertising strategies in informing consumers about their products (Butters, 1977; see Bagwell, 2007, for an excellent survey). In terms of modeling, Grossman and Shapiro (1984) is particularly related. In their model, two firms compete with each other on a Hotelling line, and firms need to send costly advertisements to consumers in order to make them informed about their products. Their focus is on firms' advertising strategy and its impact on pricing and welfare.

There is a large literature on learning through word of mouth. For instance, Smallwood and

[4] In Campbell's (2019) model this effect is absent as firms compete for fully informed consumers only.

Conlisk (1979) consider a model in which new consumers, when selecting a new product, sample the products used by the existing consumers in the population and mimic their choices; and they study how firms' current market shares affect long-run adoption among consumers. Ellison and Fudenberg (1995) and Banerjee and Fudenberg (2004) study how word-of-mouth learning affects agents' choice between alternatives with stochastic payoffs in non-market environments.[5] Overall, in this literature how the structure of the social network affects the market outcomes received little attention.

Our paper is also related to a growing literature on industrial organization which studies firm behavior when either information is diffused through a social network or there is consumption externality between neighbors (see Bloch, 2016, for a survey). Some papers consider monopoly pricing (Bloch and Querou, 2013; Campbell, 2013; Fainmesser and Galeotti, 2016), some consider monopoly advertising/seeding (Galeotti and Goyal, 2009; Campbell et al., 2017), and some study oligopoly advertising/seeding (Bimpikis et al., 2016; Goyal et al., 2019). For papers studying oligopoly pricing, Aoyagi (2018), Chen et al. (2018), and Fainmesser and Galeotti (2020) consider models in which there are direct consumption externalities between neighboring consumers and firms are able to price discriminate based on consumers' network positions. Different from those models, in our model there is no consumption externality and we focus on the diffusion of product information via a social network. Galeotti (2010) develops a duopoly model in which firms produce a homogeneous good and consumers can get informed about prices by two channels: costly search and learning from friends. He shows how the relative costs of the two channels determine pricing and welfare in equilibrium. Similarly, Kovac and Schmidt (2014) characterize market share dynamics in a Bertrand model with homogeneous products in which consumers learn about firms' prices from friends and firms play mixed pricing strategies. Different from these two models in which products are homogeneous and consumers learn prices from their friends, in our setting firms produce differentiated products and consumers learn about the existence of available products from their friends.

More recently, Campbell et al. (2020) developed a model in which consumers learn about the quality of experience goods from their friends. Their goal is to study how the structure of social networks affects the provision of quality in the long run. Their model and ours share some similar features. For instance, consumers are modelled as overlapping generations with the old generation serving as the source of information, and both papers focus on steady state. The main difference lies in different focus. Specifically, their paper focuses on learning about qualities and does not consider price competition.[6] In contrast, our paper focuses on learning about the existence of available products and its impact on price competition.

[5] Other dynamic pricing models with word-of-mouth learning include Rob and Fishman (2005) and Bergemann and Valimaki (2006).

[6] In their model all firms charge an exogenously fixed price and there is free entry of firms.

The rest of the paper is organized as follows. Section 2 sets up the model. The basic model with random connections is analyzed in Section 3. Section 4 studies the extension with homophily and Section 5 contains concluding remarks. All the proofs are relegated into the Appendix.

2 Model

Time is discrete and denoted as T, and the horizon is infinite. There are two infinitely lived firms, firms 1 and 2, competing with each other in a Hotelling model; they are located at the two end points (firm 1 at location 0). In each period, there is a new generation of consumers active in the market, and they exit the market after one period. We index consumers by generation T. Each generation of consumers is of measure 1, and they are uniformly distributed on $[0, 1]$, with a consumer's location indexed by $x \in [0, 1]$. Each consumer has a unit demand. A consumer at location x gets utility $V + \Delta - tx$ from buying firm 1's product, and $V - t(1 - x)$ from buying firm 2's product. Here t is the per-unit transportation cost, and $\Delta \geq 0$ represents firm 1's advantage over firm 2. The marginal cost of each firm is normalized to 0. We assume V is large enough so that each consumer will buy exactly one product, and the question is which one. Finally, we assume that $\Delta < t$, which ensures that both firms are active in the market.

Next we introduce consumers' information status. For each generation T, a $\lambda \in (0, 1)$ proportion of consumers, independent of location x, are aware of both products through an exogenous process (for instance, through consumer search or firms' advertising campaigns).[7] The remaining $1-\lambda$ proportion of consumers, which we call *endogenous consumers*, are initially unaware of either products. They learn about the existence of a product from their friends of the previous generation $T-1$ (old consumers who already purchased). The pattern of the social connections or friendship network is described by a distribution $\{p_k\}$, where p_k is a consumer's probability of having k friends of the old generation. We assume that $p_0 = 0$, which means that each consumer has at least one friend, thus is informed of at least one product. By having a friend of the old generation, a new consumer becomes informed of the product purchased by that friend in the last period (but not the other product). If a new consumer has two friends of the old generation who purchased different products, then the new consumer becomes aware of both products (i.e., fully informed). If all friends of a new consumer bought the same product, say product 1, then the new consumer is aware of product 1 only.[8]

[7] A positive fraction of exogenously fully informed consumers is needed to ensure that the steady-state market shares are non-degenerate (both firms have positive market shares). See footnote 11 for details.

[8] We want to point out that the feature of overlapping generations on the consumers' side is a convenient modeling device. More realistically, in each period T a fraction of new consumers arrive at the market to buy products, and they consult their friends who already bought the products in earlier periods (old consumers). More generally, all old consumers (not necessarily the old consumers who arrived at the market in period $T-1$)

In the basic model, we will focus on the case of *random connections*. That is, the friends of a new consumer are uniformly drawn at random in terms of locations, independent of the new consumer's own location. We then study *homophily* in an extension, in which case a consumer and her friends are more likely to have similar locations. Denote $\psi_T(x)$ as the probability that a consumer of generation T at location x buys from firm 1, and ψ_T as the proportion of consumers of generation T who buy product 1. That is, firm 1's market share is ψ_T and that of firm 2 is $1 - \psi_T$ in period T.

Firms know the structure of the network, but does not observe the locations of individual consumers. In the very beginning (say period 0), firms set prices P_1 and P_2 simultaneously, which remain fixed in all later periods.[9] Given the friendship network across generations, consumers' information status about the products and the market share ψ_T, in general, will evolve across periods through learning from friends. We will focus on the *steady-state* (or long-term) market share, which satisfies $\psi_{T+1} = \psi_T \equiv \psi$. In short, given P_1 and P_2, through learning from friends the market share will eventually reach a steady state. We assume that each firm's objective is to maximize its steady-state profit.[10]

Our model is closely related to Campbell (2019), with two main differences. First, Campbell (2019) considers symmetric firms ($\Delta = 0$) only, while we also consider asymmetric firms. Second and more importantly, in Campbell (2019) firms compete with each other in one period only, and the information status of the old-generation consumers is exogenously given. In contrast, in our model firms compete over time and the information status of each generation is endogenously derived. In terms of the technical analysis of the steady state, our model is also related to Campbell et al. (2019), which study how the structure of the social media network affects the prevalence of different types of media content. They also focus on steady state, under which the frequency that each type of message is forwarded remains unchanged over time. However, the research questions that our paper addresses are very different from their paper's.

To facilitate later comparisons, here we compute the equilibrium outcome in the standard Hotelling model ($\lambda = 1$ in our setting) as a benchmark. Using superscript H to denote the Hotelling outcomes, the equilibrium prices are $P_1^H = t + \frac{\Delta}{3}$ and $P_2^H = t - \frac{\Delta}{3}$, and the equilibrium market share of firm 1 is $\widehat{x}^H = \frac{1}{2} + \frac{\Delta}{6t}$.

could be consulted by new consumers. This general setting will be discussed in Section 5.

[9] This assumption will be discussed further in Section 5.

[10] One justification for this assumption is that firms are very patient. Specifically, firm 1's average per-period profit (firm 2's is similar) can be written as $\pi_1(P_1, P_2) = (1-\delta)\sum_{T=1}^{\infty} \delta^{T-1} P_1 \psi_T$, where δ is the discount factor. When $\delta \to 1$, $\pi_1(P_1, P_2)$ converges to $P_1\psi$, the steady-state per-period profit. When firms are not very patient, this assumption is still reasonable if, given any P_1 and P_2, the steady state is reached rather quickly.

3 Random Connections

In this section we study the case of random connections. First observe that for each generation, a consumer's information status can be one of the following: fully informed, aware of product 1 only, or aware of product 2 only. With random connections, a consumer's information status is independent of her location x. Our first step is to derive the steady state market share as a function of P_1 and P_2. For this purpose, we need to trace the distribution of consumers' information status across periods. Let $\phi_{F,T}$ be the proportion of consumers of generation T who are fully informed, and $\phi_{i,T}$, $i = 1, 2$, be the proportion of consumers of generation T who are aware of product i only.

Define $\widehat{x}$ as the cutoff consumer (in terms of location) who is fully informed and indifferent between product 1 and 2. Since consumers live for one period only, this $\widehat{x}$ coincides with the cutoff in the Hotelling model:

$$\widehat{x} = \frac{1}{2} + \frac{P_2 - P_1 + \Delta}{2t}. \tag{1}$$

Since firm 1 has an advantage with $\Delta \geq 0$, in equilibrium, $\widehat{x} \in [1/2, 1)$. Note that $\widehat{x}$ is also firm 1's market share if all consumers were fully informed. Given ψ_T, $\widehat{x}$, and λ, we have the following transition equations:

$$\begin{aligned}
\phi_{1,T+1} &= (1-\lambda)\sum_k p_k \psi_T^k,\ \ \phi_{2,T+1} = (1-\lambda)\sum_k p_k (1-\psi_T)^k, \\
\phi_{F,T+1} &= 1 - \phi_{1,T+1} - \phi_{2,T+1} = \lambda + (1-\lambda)[1 - \sum_k p_k[\psi_T^k + (1-\psi_T)^k], \\
\psi_{T+1} &= \phi_{1,T+1} + \widehat{x}\phi_{F,T+1}.
\end{aligned}$$

In the first (second) equation, the consumers in generation $T+1$ who are informed of product 1 (2) only must be someone: (i) who are not exogenously fully informed, and (ii) whose friends of generation T all bought product 1 (2) only. In the third equation, the fully informed consumers in generation $T+1$ include consumers who are exogenously fully informed and endogenous consumers whose friends of generation T bought different products. In the last equation, the fraction of consumers buying product 1 in generation $T+1$ consists of two groups: consumers who are aware of product 1 only, and the fully informed consumers who prefer product 1.

Combining the above equations, we have the following transition equation, which specifies ψ_{T+1} as a function of ψ_T:

$$\psi_{T+1} = \widehat{x} + (1-\lambda)\sum_k p_k[(1-\widehat{x})(\psi_T)^k - \widehat{x}(1-\psi_T)^k] \equiv H(\psi_T). \tag{2}$$

The three terms of $H(\psi_T)$ in (2) can be interpreted as follows. The first term $\widehat{x}$ is firm 1's full-information market share. The second term is the fraction of consumers who "wrongly" bought product 1 (they prefer firm 2's product but are informed of product 1 only); firm 1 gains this portion of market share relative to the full-information benchmark. Similarly, the third term is the fraction of consumers who "wrongly" bought product 2; and firm 1 loses this portion of market share relative to the full-information benchmark. Taken together the second and third terms, it represents firm 1's net gain of market share among the partially informed consumers relative to the full-information benchmark.

In the steady state, $\psi_{T+1} = \psi_T \equiv \psi$; that is, $\psi = H(\psi)$. By (2), the steady-state equation can be explicitly written as

$$\psi = \widehat{x} + (1-\lambda)\sum_k p_k[(1-\widehat{x})\psi^k - \widehat{x}(1-\psi)^k] \equiv H(\psi). \tag{3}$$

Note that P_1 and P_2 affect the steady-state market share ψ only through their effects on $\widehat{x}$. The next lemma shows that $\widehat{x}$ induces a unique steady-state market share ψ.

Lemma 1 *Given $\widehat{x} \in [1/2, 1)$, λ, and $\{p_k\}$, there is a unique steady-state ψ, which is globally stable and satisfies $\psi \in [\widehat{x}, 1)$. Moreover, the steady-state ψ is strictly increasing in $\widehat{x}$.*

The result that $\psi \geq \widehat{x}$ is intuitive. To see this, suppose firm 1's market share equals $\widehat{x} \geq 1/2$. Then, due to learning from friends, there will be (weakly) more consumers who wrongly purchase product 1 than those who wrongly purchase product 2, since firm 1 has a larger full-information market share. This indicates that firm 1's steady-state market share ψ would be (weakly) bigger than $\widehat{x}$.

The relationship between ψ and $\widehat{x}$ depends on the network structure $\{p_k\}$. To make the relationship more transparent, we will pay special attention to regular networks under which $\{p_k\}$ is degenerate. In particular, we define a *k-friend network* as a regular network under which every consumer has exactly k friends ($p_k = 1$ for some $k \geq 1$). Among *k-friend networks* there are two extreme networks. The first one is the *single-friend network*, under which each consumer has exactly one friend ($p_1 = 1$). The second one is the *infinite-friend network*, under which each consumer has infinitely many friends ($k \to \infty$). We use the term generic networks for networks other than the single-friend network and the infinite-friend network.[11] The next lemma sheds some light on the relationship between ψ and $\widehat{x}$ under different networks.

Lemma 2 *(i) Under the single-friend network or the infinite-friend network, $\psi = \widehat{x}$. (ii) Under generic networks, $\psi > \widehat{x}$ if $\widehat{x} > 1/2$. (iii) Under k-friend networks, fixing $\widehat{x} \in (1/2, 1)$, ψ strictly decreases in k if $k \geq 2$.*

Lemma 2 holds the key in understanding later results. The result regarding the infinite-friend network is quite intuitive: under this network, all consumers in each generation are fully

[11] Under any generic network, if $\lambda = 0$, then by (3) it can be verified that $\psi = 1$ whenever $\widehat{x} \in (\frac{1}{2}, 1)$.

informed, and therefore the steady-state market share ψ equals to the full-information one, $\widehat{x}$. Under the single-friend network, when $\psi = \widehat{x}$, the fraction of endogenous consumers who wrongly buy product 1 exactly equals to those who wrongly buy product 2, both of which are $(1-\widehat{x})\widehat{x}$. Thus firm 1's steady-state market share is $\widehat{x}$.[12] For *k-friend networks with* $k \geq 2$, as k increases (the network becomes more connected), the total number of partially informed consumers ($(1-\lambda)[\psi^k + (1-\psi)^k]$) decreases, but the ratio of the number of consumers informed of product 1 only to those informed of product 2 only, $(\frac{\psi}{1-\psi})^k$, increases because $\psi > \widehat{x} > 1/2$. The first effect tends to decrease firm 1's steady-state market share, and the second effect works in the opposite direction. It turns out that when $k \geq 2$, the first effect dominates, and firm 1's net gain of market share among the partially informed consumers decreases in k. Therefore, among k-friend networks, the two-friend network leads to the biggest steady-state market share for firm 1.

We are also interested in the curvature of $\psi(\widehat{x})$, as it will be useful for establishing the existence and uniqueness of equilibrium.

Lemma 3 *(i) For any* $\{p_k\}$, $\frac{d^2\psi}{d\widehat{x}^2}|_{\widehat{x}=1/2} = 0$ *and* $\lim_{\lambda\to 1} \frac{d^2\psi}{d\widehat{x}^2} = 0$. *(ii) Under well-connected networks (*$p_k = 0$ *for all* $k \leq \overline{k}$ *and* $\overline{k}$ *is large),* $\frac{d^2\psi}{d\widehat{x}^2} \to 0$. *(iii) Under sparsely-linked networks (*$p_k = 0$ *for* $k \geq 4$*, or each consumer has at most* 3 *friends),* $\frac{d^2\psi}{d\widehat{x}^2} \geq 0$ *when* $\widehat{x} \leq 1/2$ *and* $\frac{d^2\psi}{d\widehat{x}^2} \leq 0$ *when* $\widehat{x} \geq 1/2$. *(iv) Under any generic network* $\{p_k\}$*, if* λ *is big enough, then* $\frac{d^2\psi}{d\widehat{x}^2} \geq 0$ *when* $\widehat{x} \leq 1/2$ *and* $\frac{d^2\psi}{d\widehat{x}^2} \leq 0$ *when* $\widehat{x} \geq 1/2$.

Lemma 3 specifies a set of sufficient conditions for $\psi(\widehat{x})$ to be convex when $\widehat{x} \leq 1/2$ and concave when $\widehat{x} \geq 1/2$.[13] Note that the curvature of $\psi(\widehat{x})$ only depends on $\{p_k\}$ and λ. In the remaining of this section we will focus on the set of $\{p_k\}$ and λ such that $\psi(\widehat{x})$ satisfies the above convexity/concavity property.

Now we are ready to characterize steady-state equilibria. Firm 1's and firm 2's profits in steady state are $\pi_1(P_1, P_2) = P_1\psi(\widehat{x})$ and $\pi_2(P_1, P_2) = P_2[1-\psi(\widehat{x})]$, respectively. Each firm i chooses P_i, given P_j, to maximizes its steady state profit. The first-order conditions yield (with P_i^e being the equilibrium prices)

$$P_1^e = \frac{2t\psi}{\frac{d\psi}{d\widehat{x}}} \text{ and } P_2^e = \frac{2t(1-\psi)}{\frac{d\psi}{d\widehat{x}}}, \tag{4}$$

[12] Specifically, for k-friend networks,

$$H(\widehat{x}) - \widehat{x} = (1-\lambda)(1-\widehat{x})\widehat{x}[\widehat{x}^{k-1} - (1-\widehat{x})^{k-1}].$$

[13] For part (iv) to hold, λ could be relatively small. For example, when $\lambda = 1/4$, $\psi(\widehat{x})$ has the desired convexity/concavity property under the network $p_1 = p_2 = p_3 = 1/3$, and under the network $p_1 = p_2 = p_3 = p_4 = p_5 = 1/5$.

where, by (3),

$$\frac{d\psi}{d\widehat{x}}=\frac{1-(1-\lambda)\sum_k p_k[(\psi)^k+(1-\psi)^k]}{1-(1-\lambda)\sum_k kp_k[(1-\widehat{x})(\psi)^{k-1}+\widehat{x}(1-\psi)^{k-1}]}. \tag{5}$$

Notice that $d\psi/d\widehat{x}>0$ (by Lemma 1) captures the sensitivity of ψ to $\widehat{x}$ and largely determines the intensity of competition. Combining (1), (4), and (5), we have the following pricing equation:

$$\widehat{x}=\frac{1}{2}+\frac{\Delta}{2t}-(2\psi-1)\frac{1-(1-\lambda)\sum_k kp_k[(1-\widehat{x})(\psi)^{k-1}+\widehat{x}(1-\psi)^{k-1}]}{1-(1-\lambda)\sum_k p_k[(\psi)^k+(1-\psi)^k]}. \tag{6}$$

Equations (3) and (6), with two unknowns ($\widehat{x}$ and ψ), jointly determine steady-state equilibria. Denote $(\widehat{x}_e,\psi_e)$ as an equilibrium pair of $(\widehat{x},\psi)$.[14]

Proposition 1 *There is a unique candidate equilibrium (satisfying the steady-state equation and the first-order conditions), which satisfies* $\widehat{x}_e\in[\frac{1}{2},\frac{1}{2}+\frac{\Delta}{2t}]$*. If* $\Delta=0$*, the candidate equilibrium is symmetric with* $\widehat{x}_e=\psi_e=1/2$*. The candidate equilibrium is an equilibrium under either of the following conditions: (i)* $\{p_k\}$ *is the single-friend network or a well-connected network; (ii)* λ *is relatively large.*

The conditions (i) and (ii) specified in Proposition 1 are sufficient conditions under which the second-order conditions are satisfied globally. Intuitively, under both conditions in the limit the model converges to the Hotelling benchmark. These two conditions are far from being necessary. Even when λ is relatively small so that the second-order conditions are not satisfied globally, each firm's profit function could still be single-peaked, meaning that the candidate equilibrium satisfying the first-order conditions is indeed the equilibrium.[15]

3.1 Symmetric firms

We first consider symmetric firms with $\Delta=0$. Since Campbell (2019) focuses solely on symmetric firms, the comparison in this subsection will reveal clearly how introducing steady state (or long-term) demand affects results. With $\Delta=0$, the equilibrium is symmetric with $\widehat{x}_e=\psi_e=1/2$, $P_1^e=P_2^e\equiv P^e=t/(\frac{d\psi}{d\widehat{x}}|_{\widehat{x}=1/2})$, and (5) becomes

$$\frac{d\psi}{d\widehat{x}}|_{\widehat{x}=1/2}=\frac{1-(1-\lambda)\sum_k p_k(\frac{1}{2})^{k-1}}{1-(1-\lambda)\sum_k kp_k(\frac{1}{2})^{k-1}}. \tag{7}$$

[14] From $(\widehat{x}_e,\psi_e)$, we can recover the equilibrium prices P_1^e and P_2^e based on (4) and (5).

[15] For instance, consider the following examples. Network 1 has $p_1=p_2=p_3=1/3$ and network 2 has $p_2=p_3=p_4=p_5=1/4$. The other parameter values are $\lambda=1/4$, $t=1$, and Δ can be either 0 or 1/4. For each case, we plot firm i's profit function $\pi_i(P_i,P_{je})$ (the other firm's price is fixed at the equilibrium price P_{je}). In all cases (four cases in total), each firm's profit function is single-peaked at P_{ie}, though the profit function is convex in P_i when P_i is large.

The expression of the equilibrium price reveals that the intensity of competition depends on the sensitivity of ψ to $\widehat{x}$, evaluated at $\widehat{x}_e$.

To examine how the network structure affects the equilibrium price, we define connectivity in terms of first-order stochastic dominance (FOSD): a network $\{p''_k\}$ is more connected than $\{p'_k\}$ if $\{p''_k\}$ FOSD $\{p'_k\}$.

Proposition 2 *With symmetric firms, the equilibrium price is not monotonic in network connectivity. In particular, the following results hold. (i) Under either the single-friend network or the infinite-friend network, $P^e = t$, but under any other generic network, $P^e < t$. (ii) Among the k-friend networks, the equilibrium price is the lowest under the 2-friend network; when k increases from 1 to 2, the equilibrium price decreases; for $k \geq 2$, the equilibrium price strictly increases in k.*

The non-monotonicity result in Proposition 2 can be understood in light of part (iii) of Lemma 2. In more general terms, we can decompose the net effect of network connectivity on $d\psi/d\widehat{x}$ into two effects: the *volume effect* and the *ratio effect*. Specifically, as the network becomes more connected, more consumers are fully informed and the fraction of partially informed consumers decreases. This volume effect tends to reduce the sensitivity of ψ to $\widehat{x}$ and soften competition, as there are fewer partially informed consumers to compete for. On the other hand, a more connected network means that, for the same (non-equilibrium) market share $\psi > 1/2$, the ratio of the number of consumers informed of product 1 only to those informed of product 2 only will increase.[16] This means that firm 1 can get a bigger market share among the partially informed consumers through dynamic social learning. This ratio effect tends to increase the sensitivity of ψ to $\widehat{x}$ and intensify competition. Since the volume effect and the ratio effect work in opposite directions, the equilibrium price is not monotonic in network connectivity. Among the k-friend networks, it turns out that the ratio effect dominates when k increases from 1 to 2, and the volume effect dominates when $k \geq 2$.

Proposition 2 is qualitatively different from the corresponding result in Campbell (2019), where the equilibrium price monotonically decreases as the friendship network becomes more connected. The main reason for the difference is that in Campbell's (2019) one-period model, firms compete for fully informed consumers only. As the network becomes more connected, since the fraction of fully informed consumers increases, competition intensifies and the price decreases. In contrast, in our model firms are competing not only for fully informed consumers in the current period, but also for partially informed consumers in future periods. A bigger market share of firm i today (achieved by setting a lower price), through the friendship network, will lead to more (fewer) partially informed consumers who are informed of product i (j) only

[16] As mentioned earlier, under k-friend networks this ratio is $(\frac{\psi}{1-\psi})^k$, which is increasing in k when $\psi > 1/2$.

in the next period, which translates to an even bigger market share in the future. This dynamic channel is absent in Campbell (2019).

In a duopoly model with homogeneous goods, Galeotti (2010) also shows that the average equilibrium price is non-monotonic in network connectivity. As mentioned earlier, in his model consumers can get fully informed through costly search and learning from friends. As consumers have more friends, its direct effect is to increase the fraction of fully informed consumers, given any fixed level of search effort. However, it also crowds out consumers' search effort as the free-riding problem becomes more severe, which tends to reduce the fraction of fully informed consumers. These two opposite effects lead to the non-monotonicity of price in network connectivity.[17] The non-monotonicity result in our paper is due to a different mechanism: an increase in connectivity leads to the volume effect and the ratio effect about the partially informed consumers, and the combination of these two opposite effects generates non-monotonicity. Another qualitative difference is that in Galeotti (2010) the relationship between price and network connectivity exhibits an inverse U shape (the price reaches the maximum for some intermediate level of connectivity), while in our model the relationship is U shape (the price reaches the minimum for an intermediate level of connectivity). This is because in his (static) model firms compete for fully informed consumers only, while in our model firms also compete for partially informed consumers due to the dynamic learning channel.

To capture more precisely the dynamic learning effect, in the symmetric equilibrium we compute the sensitivity of firm 1's current period demand to its price, $|\frac{\partial Q_1}{\partial P_1}|$, and the sensitivity of firm 1's steady-state (long-run) demand to its price, $|\frac{\partial \psi}{\partial P_1}|$. In particular,

$$|\frac{\partial Q_1}{\partial P_1}| = \frac{1-(1-\lambda)\sum_k p_k(\frac{1}{2})^{k-1}}{2t},$$

where the numerator is precisely the fraction of fully informed consumers, as in the current period only those consumers are sensitive to prices. Using (7), we have

$$|\frac{\partial \psi}{\partial P_1}| = \frac{|\frac{\partial Q_1}{\partial P_1}|}{1-(1-\lambda)\sum_k kp_k(\frac{1}{2})^{k-1}} > |\frac{\partial Q_1}{\partial P_1}|.$$

Observe that the long-run demand is more sensitive to price than the current-period demand is, and the difference between the two sensitivities exactly captures the dynamic learning effect. This also suggests that using the sensitivity of current-period demand to price in empirical studies to fit firms' behaviors could be misleading. In the real world, what firms try to maximize

[17] In a related paper on monopoly advertising/seeding, Galeotti and Goyal (2009) show that the relationship between the level of advertising and network connectivity is also non-monotonic. The reason is that advertising and word of mouth (WOM) can be substitutes (WOM means that the firm can advertise less to reach the same number of consumers) or complements (WOM makes advertising more effective), depending on the cost of advertising.

is more likely to be long-run profits rather than short-run profits, and thus the sensitivity of long-run demand to price is a more appropriate measure. In other words, using the sensitivity of current-period demand would underestimate the sensitivity of the (long-run) demand that firms actually care about. Our model also provides a justification for firms' seemingly puzzling overemphasis on current market shares over the current profits.[18] This is because, due to the dynamic learning effect, a firm's current market share also affects the evolution of its future market share and thus its long-run profits.[19]

Proposition 3 *Suppose firms are symmetric and networks are generic. (i) If $\{p''_k\}$ FOSD $\{p'_k\}$ and $p''_1 = p'_1$, then the equilibrium price is higher under $\{p''_k\}$ than under $\{p'_k\}$. (ii) The equilibrium price is increasing in λ, the fraction of exogenously fully informed consumers.*

For generic (non-regular) networks, the relationship between network connectivity and equilibrium price follows a pattern similar to the one under regular (the k-friend) networks. Part (i) of Proposition 3 shows that, if a FOSD change in connectivity does not reduce p_1 (thus puts more probabilities on higher number of links), then it softens competition and increases the equilibrium price. However, if a FOSD change reduces p_1, then it may intensify competition and reduce prices (see Example 1). For the same reason, the equilibrium price is not monotonic with respect to changes of mean-preserving spread (see Example 2).[20]

Part (ii) of Proposition 3 is a surprising result. In standard models of price competition (such as the search models of Varian (1980) and Stahl (1989)), an increase in the fraction of exogenously fully informed consumers typically intensifies competition and reduces prices. By contrast, in our model the result is the opposite. The underlying reason, as mentioned earlier, is that firms in our model also compete for partially informed consumers of future generations through the dynamic social learning. When λ increases, there are fewer partially informed consumers to compete for, or the steady-state demand becomes less sensitive to prices. As a result, competition is softened and prices increase.

Example 1 *(FOSD). Let $\lambda = 1/4$, $\{p'_k\} = \{p'_1 = p'_2 = 1/2\}$, and $\{p''_k\} = \{p''_1 = p''_2 = p''_3 = 1/3\}$. Note that $\{p''_k\}$ FOSD $\{p'_k\}$. The values of $\frac{d\psi}{d\widehat{x}}|_{\widehat{x}=1/2}$ under $\{p'_k\}$ and $\{p''_k\}$ are $7/4$ and $9/5$, respectively, leading to a lower equilibrium price under $\{p''_k\}$.*

[18] According to Farris et al. (2010), 67% of senior marketing managers and executives regard market share as an essential performance indicator in itself. Anecdotal evidence also suggests that many business leaders target market share (instead of profit) when setting business strategies (Edeling and Himme (2018)).

[19] Our explanation is different from Bendle and Vandenbosch (2014), which explains why competitor orientation can persist and even thrive based on evolutionary games. Our explanation is also consistent with the empirical finding that the market share-financial performance elasticity is larger in industries where learning from friends plays an important role, such as service industries in emerging markets and B2C industries, as documented in Edeling and Himme (2018).

[20] In Campbell (2019), a mean-preserving spread in $\{p_k\}$ always softens competition and increases prices.

Example 2 *(Mean-preserving spread). Let* $\lambda = 1/4$, $\{p_k\}_A = \{p_2 = 1\}$, *and* $\{p_k\}_B = \{p_1 = p_2 = p_3 = 1/3\}$, $\{p_k\}_C = \{p_4 = 1\}$, *and* $\{p_k\}_D = \{p_3 = p_5 = 1/2\}$. *Note that* $\{p_k\}_B$ *is a mean-preserving spread of* $\{p_k\}_A$, *and* $\{p_k\}_D$ *is also a mean-preserving spread of* $\{p_k\}_C$. *The values of* $\frac{d\psi}{d\hat{x}}|_{\hat{x}=1/2}$ *under* $\{p_k\}_A$ *and* $\{p_k\}_B$ *are* $5/2$ *and* $9/5$, *respectively; thus the equilibrium price is higher under* $\{p_k\}_B$ *than under* $\{p_k\}_A$. *The values of* $\frac{d\psi}{d\hat{x}}|_{\hat{x}=1/2}$ *under* $\{p_k\}_C$ *and* $\{p_k\}_D$ *are* 1.45 *and* 1.47, *respectively; thus the equilibrium price is lower under* $\{p_k\}_D$ *than under* $\{p_k\}_C$.

Next we consider the impact of network connectivity on social welfare. Denote the total welfare and consumer surplus in the steady-state equilibrium as W and CS, respectively. In particular,

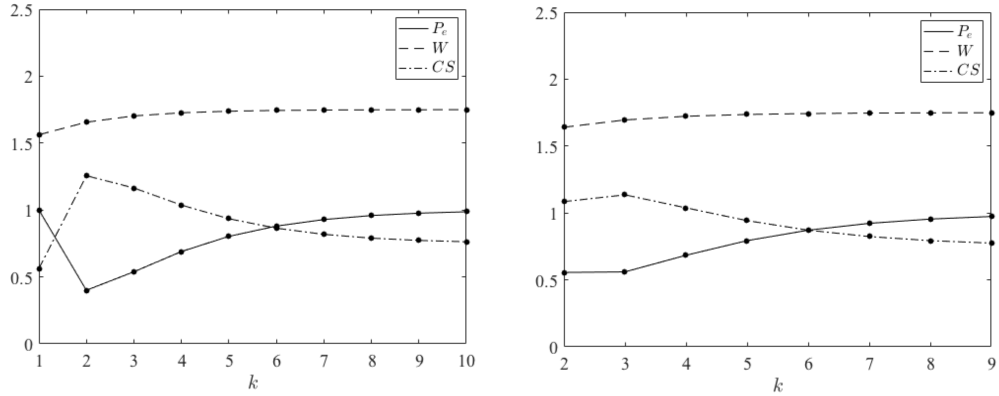
$$\begin{aligned} W &= V - 2\{\frac{t}{4}[\frac{1}{2} - (1-\lambda)\sum_k p_k(\frac{1}{2})^{k+1}] + \frac{3}{4}t(1-\lambda)\sum_k p_k(\frac{1}{2})^{k+1}\} \\ &= V - \frac{t}{4} - t(1-\lambda)\sum_k p_k(\frac{1}{2})^{k+1}. \end{aligned} \tag{8}$$

Total welfare W equals to V minus the total transportation costs incurred. The latter includes the transportation costs incurred by the consumers who bought their right products, which is captured by the first term in the braces; among these consumers the average transportation costs per consumer is $t/4$. The second term in the braces is the total transportation costs incurred by the consumers who bought wrong products; among these consumers, the average transportation costs per consumer is $3t/4$. As to consumer surplus, $CS = W - P^e$, which by (8) can be written as

$$CS = V - \frac{t}{4} - t\frac{1-(1-\lambda)\sum_k kp_k(\frac{1}{2})^{k-1}}{1-(1-\lambda)\sum_k p_k(\frac{1}{2})^{k-1}} - t(1-\lambda)\sum_k p_k(\frac{1}{2})^{k+1}. \tag{9}$$

Proposition 4 *With symmetric firms, total welfare* W *is increasing in the connectivity of* $\{p_k\}$. *However, consumer surplus* CS *is not monotonic as the connectivity of* $\{p_k\}$ *increases. In particular, among the k-friend networks, (i)* CS *is higher under the two-friend network than under the single-friend network; (ii) when* $k \geq 3$, CS *is decreasing in* k*; (iii) when* k *increases from* 2 *to* 3, CS *decreases if* $\lambda \leq \sqrt{113} - 10 \simeq 0.63$, *and increases otherwise.*

The result that total welfare always increases in network connectivity is easy to understand. As the network becomes more connected, more consumers are fully informed and thus fewer consumers buy wrong products, which reduces the total transportation costs and increases total welfare. As to consumer surplus, besides the *information effect* mentioned above, there is also a *pricing effect* as the equilibrium price changes with network connectivity. Since the pricing effect is not monotonic, the overall effect of network connectivity on consumer surplus



(a): regular networks (b): networks with three-point distribution

Figure 1: Comparative Statics

is not monotonic either. Among the k-friend networks, as k changes from 1 to 2, the price decreases; thus the pricing effect and the information effect work in the same direction and CS increases. With $k \geq 2$, the price is increasing in k, which means that the pricing effect and the information effect work in opposite directions. It turns out that when $k \geq 3$ the pricing effect always dominates so that CS is decreasing in k. Intuitively, when the network is already relatively well connected, the fraction of consumers who buy wrong products is already small. Thus a further increase in connectivity only leads to a small decrease in the number of consumers buying wrong products, and hence the information effect is small. On the other hand, a price increase induced by an increase in connectivity hurts all consumers. Therefore, the pricing effect dominates when k is relatively large.

Comparing our predictions and those of Campbell (2019), while the result regarding total welfare is the same, the predictions on consumer surplus are quite different, as the pricing effects across two models are qualitatively different. In particular, Campbell (2019) predicts that consumer surplus always increases in connectivity, while our model predicts that it increases in connectivity when the network is sparsely connected, but decreases in connectivity otherwise.

To see the magnitude of the effect of network connectivity on the equilibrium price, social welfare and consumer surplus, consider the following examples. We define networks $\{p_k\}$ with $p_{k-1} = p_k = p_{k+1} = 1/3$ as networks with three-point distributions, which is indexed by k. With $t = 1$, $\lambda = 1/4$, and $V = 2$, Figure 1 illustrates how the equilibrium price and welfare change with k among the k-friend networks and networks with three-point distributions. For the k-friend networks, the price reduction and the increase in CS between the single-friend network and the two-friend network are significant. When k is between 2 and 6, as k increases the price increases considerably and CS decreases considerably. When k is bigger than 7, both the price increase and CS reduction become relatively insignificant as k increases. This example shows that, among not very well connected regular networks, an increase in connectivity could have quantitatively significant impacts on both price and welfare. A similar pattern holds under the networks with three-point distributions, though the changes in price and CS become smaller (relative to the regular networks) as k increases. Overall, the two examples indicate that the dynamic learning effect is quantitatively important under not very well connected networks.

3.2 Asymmetric firms

Next we consider asymmetric firms with $0 < \Delta \leq t$. In this scenario, firms charge different prices and have different market shares in equilibrium.

Proposition 5 *(i) The single-friend network and the infinite-friend network lead to the same equilibrium prices and market shares, which coincide with those in the standard Hotelling model. That is,*

$$P_1^e = P_1^H = t + \frac{\Delta}{3},\ P_2^e = P_2^H = t - \frac{\Delta}{3},\ \text{and}\ \psi_e = \widehat{x}^H = \frac{1}{2} + \frac{\Delta}{6t}.$$

(ii) Under generic networks, $\frac{d\psi_e}{d\widehat{x}}|_{\widehat{x}_e} > 1$*. Moreover, compared to the Hotelling benchmark, the equilibrium market share of firm 1 (2) is higher (lower), and firm 2's equilibrium price is lower:* $\psi_e > \widehat{x}^H$*, and* $P_2^e < P_2^H$*. (iii) Under the k-friend networks with* $k \geq 2$ *and* Δ *not being too large so that* ψ_e *is not too far away from* $1/2$*,* ψ_e *decreases in k.*

Proposition 5 shows that under generic networks, relative to the Hotelling benchmark, with learning from friends competition is more intense (reflected in $\frac{d\psi_e}{d\widehat{x}}|_{\widehat{x}_e} > 1$ and a lower equilibrium price for firm 2), the equilibrium market share of the advantaged firm is larger, and for the less advantaged firm its price, market share, and hence profit are all lower. The underlying reason is again the dynamic learning across generations. With asymmetric firms, the full-information market share of firm 1 is strictly bigger than $1/2$. Through learning from friends, firm 1 can gain additional market share among partially informed consumers. To counter the reduced market share due to the dynamic learning, firm 2 reduces its price. Proposition 5 also indicates that the market share and prices are not monotonic in network connectivity: the two extreme networks with the lowest and highest connectivity have the same market share and prices, while for all other networks the market share ψ_e is higher and P_2^e is lower. Part (iii) implies that, among the k-friend networks with $k \geq 2$ and when two firms are not too asymmetric, an increase in connectivity reduces the equilibrium market share of the advantaged firm (the intuition will be provided shortly).

We would like to examine how connectivity affects the equilibrium market share and prices under more general networks. However, since both $\widehat{x}_e$ and ψ_e vary with the network structure,

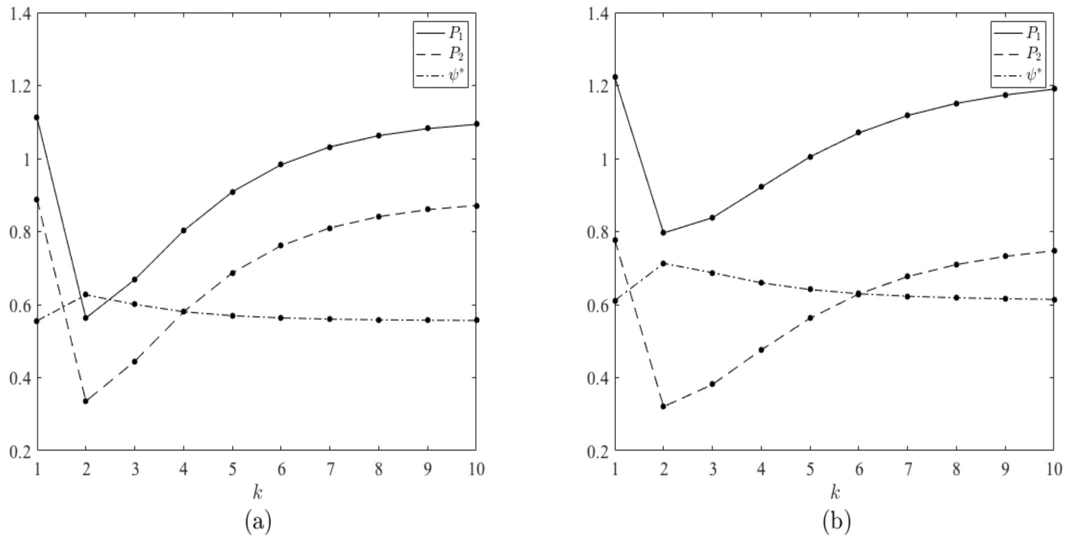


Figure 2: Prices and Market Share - the k-Friend Networks

it is hard to carry out analytical analysis.[21] Therefore, we resort to numerical simulations. With $t = 1$ and $\lambda = 1/4$, Figure 2 (3) illustrates how the equilibrium market share and prices change as k varies under the k-friend networks (networks with three-point distribution). In both figures, $\Delta = 1/3$ in panel (a) and $\Delta = 2/3$ in panel (b). First observe that in Figure 2 both panels exhibit the same pattern. As k increases from 1 to 2, firm 1's market share increases considerably and both firms' prices decrease sharply. When $k \geq 2$, as k increases firm 1's market share decreases and both firms' prices increase. Both panels in Figure 3 exhibit a similar pattern, though the changes become less sensitive to k. The underlying reason for this pattern can be found in part (iii) of Lemma 2. In particular, when $k \geq 2$, a further increase in k reduces firm 1's market share ψ for any given $\widehat{x}$. This effect tends to reduce ψ_e as k increases.[22] Note that the qualitative relationship between prices and k is the same as in the case with symmetric firms. Broadly speaking, the general pattern is that if the initial network is already relatively well connected, then a further increase in connectivity would decrease firm 1's market share and soften competition, as the dynamic learning effect is dampened.

[21] In particular, both equations (3) and (6) are highly nonlinear and contain higher order polynomials.

[22] The effect of endogenous adjustment in $\widehat{x}$, caused by the price adjustments, on ψ is secondary, since the two prices adjust in the same direction as shown in the figure.

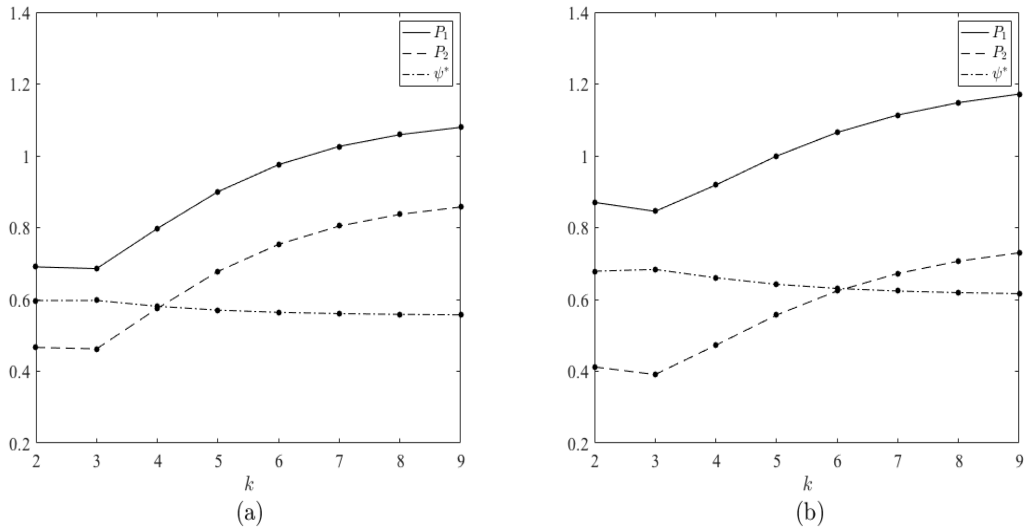


Figure 3: Prices and Market Share - Networks with Three-point Distributions

Finally, we examine the impact of network connectivity on total welfare W and consumer surplus CS. In particular,

$$\begin{aligned} W &= V + \Delta\psi_e - t[\frac{\widehat{x}_e}{2}\psi_e + \frac{(1-\widehat{x}_e)}{2}(1-\psi_e)] \\ &\quad -\frac{t}{2}(1-\lambda)\sum_k p_k[(1-\widehat{x}_e){\psi_e}^k + \widehat{x}_e(1-\psi_e)^k], \qquad (10) \\ CS &= W - P_1^e\psi_e - P_2^e(1-\psi_e). \end{aligned}$$

The last term in (10) is the additional transportation costs incurred by the consumers who buy wrong products. Again, we use numerical simulations to conduct analysis.

Using the same parameter values as in Figure 2, Figure 4 plots W and CS as k increases among the k-friend networks. Both panels exhibit the same pattern. As k increases from 1 to 2, both W and CS increase. When $k \geq 2$, as k increases CS monotonically decreases (since both firms' prices increase). Interestingly, as k increases W increases slightly initially, but when $k \geq 6$ W decreases slightly in k. This result is qualitatively different from the one under symmetric firms, where W is always increasing in k. The underlying reason is that with asymmetric firms, firm 1 (the advantaged firm) charges a higher price than firm 2 in equilibrium, which leads to a distortion in product allocation (a lower market share of firm 1 relative to the first best) and loss in total welfare.[23] When k increases, firm 1's market share advantage due to the dynamic learning effect is reduced, which means that the product misallocation due to price difference increases. This effect tends to reduce total welfare. When

[23] Efficiency in product allocation requires $P_1 = P_2$ as in the Hotelling model.

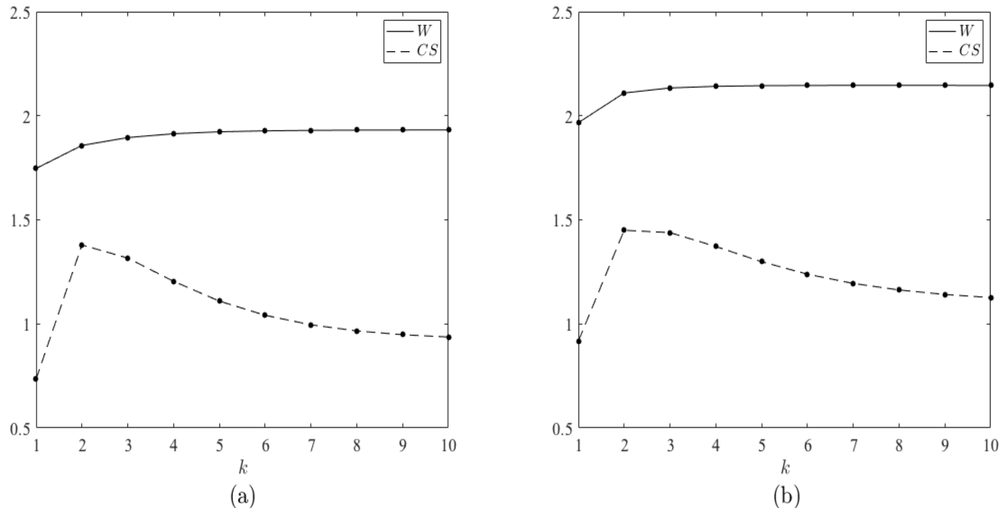


Figure 4: Welfare - the k-Friend Networks

k is relatively large, this effect can dominate the information effect (which tends to increase total welfare as k increases), and make total welfare decreasing in k.

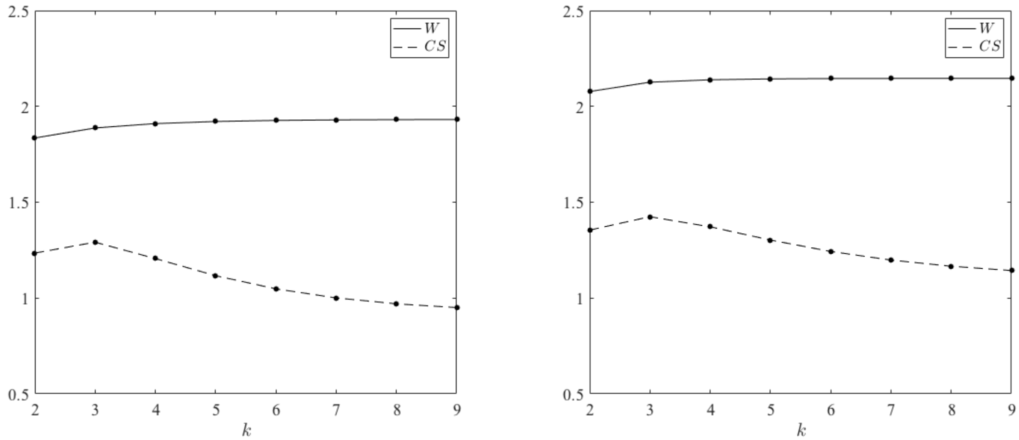
With the same parameter values as Figure 3, Figure 5 examines the impact of network connectivity on welfare among networks with three-point distributions. The pattern is almost the same as the one under the k-friend networks. This suggests that the pattern identified earlier is relatively robust.

4 Homophily

A prevalent feature of social networks is homophily. That is, individuals tend to have friends who are similar to themselves.[24] In our context, homophily is reflected in the pattern that consumers at similar locations in the product space are more likely to be friends. We use parameter $\alpha \in [0,1]$ to capture the degree of homophily. In particular, for each consumer at location x, with probability $1-\alpha$ a friend is drawn uniformly at random from location $[0,1]$, and with probability α a friend is drawn from location $[x-\delta, x+\delta]$, with $\delta \geq 0$ but very small. With this setup, a bigger α implies a higher degree of homophily. The purpose of this section is to study the impact of homophily on pricing and welfare.

With homophily, consumers become more likely to be aware of their preferred products, as their friends are more likely to have bought their preferred products. In order to trace the evolution of consumers' information status and purchasing behavior, we need to separate consumers into two types (groups). For a given $\widehat{x}$, denote L type consumers as those with $x \leq \widehat{x}$

[24] See McPherson, Smith-Lovin, and Cook (2001).



(a) (b)

Figure 5: Welfare - Networks with Three-point Distributions

(who buy product 1 if fully informed), and R type consumers as those with $x > \widehat{x}$ (who buy product 2 if fully informed). Type L consumers and type R consumers differ in information received in the ex ante sense, as their friends' purchasing behaviors are different statistically. In order to make the analysis tractable, we further assume that $\delta \to 0$. With $\delta \to 0$, the same process governs all L type consumers' information status and purchasing behaviors,[25] and the same holds among all R type consumers.

To proceed, we first introduce notations. Denote $\psi_{L,T}$ ($\psi_{R,T}$) as the the proportions of L (R) type consumers of generation T who buy product 1, $\psi_T = \widehat{x}\psi_{L,T} + (1-\widehat{x})\psi_{R,T}$ as the market share of firm 1 in period T (which is also the probability that a random consumer of generation T buys product 1), and $\phi_{jh,T}$ as the proportion of type h ($h = L, R$) consumers of generation T who are informed of product j only. The transition equations (across generations) are listed below.

$$\begin{aligned}
\phi_{1R,T+1} &= (1-\lambda)\sum_k p_k[(1-\alpha)\psi_T + \alpha\psi_{R,T}]^k, \\
\phi_{2L,T+1} &= (1-\lambda)\sum_k p_k[1-(1-\alpha)\psi_T - \alpha\psi_{L,T}]^k, \\
\psi_{L,T+1} &= 1-\phi_{2L,T+1},\ \psi_{R,T+1} = \phi_{1R,T+1}.
\end{aligned}$$

In the first equation, $(1-\alpha)\psi_T + \alpha\psi_{R,T}$ is the probability that a type R consumer's (of generation $T+1$) friend (of generation T) buys product 1, and this consumer is informed

[25] If $\delta > 0$ and does not converge to 0, then the information received (in the ex ante sense) among L (or R) type consumers depends on their locations. For instance, a type L consumer located very close to $\widehat{x}$ would receive information that is statistically different from a type L located close to 0, as the former's neighbors include some type R consumers. This feature would make the analysis intractable.

of product 1 only if all his generation T friends bought product 1. Similarly, in the second equation, $1-(1-\alpha)\psi_T-\alpha\psi_{L,T}$ is the probability that a type L consumer's (of generation $T+1$) friend (of generation T) buys product 2, and this consumer is informed of product 2 only if all his generation T friends bought product 2. The above consumers in generation $T+1$ are informed of "wrong" products only, and thus will buy "wrong" products, as indicated by the third and fourth equations.

Let ψ_L, ψ_R, and ψ be the steady-state market shares of firm 1. The steady-state conditions require $\psi_{L,T+1}=\psi_{L,T}\equiv\psi_L$ and $\psi_{R,T+1}=\psi_{R,T}\equiv\psi_R$. Using the above equations, we have the following steady-state equations:

$$1-(1-\lambda)\sum_k p_k[1-(1-\alpha)\psi-\alpha\psi_L]^k = \psi_L, \tag{11}$$

$$(1-\lambda)\sum_k p_k[(1-\alpha)\psi+\alpha\psi_R]^k = \psi_R, \tag{12}$$

$$\widehat{x}\psi_L+(1-\widehat{x})\psi_R = \psi. \tag{13}$$

The above system consists of three equations with three unknowns, ψ, ψ_L and ψ_R. We can solve these three endogenous variables as a function of $\widehat{x}$. Equations (11), (12), and (13) can also be combined as

$$\psi=\widehat{x}+(1-\lambda)\sum_k p_k\{(1-\widehat{x})[(1-\alpha)\psi+\alpha\psi_R]^k-\widehat{x}[1-(1-\alpha)\psi-\alpha\psi_L]^k\}. \tag{14}$$

As indicated by (14), ψ equals the full-information market share $\widehat{x}$ plus the fraction of consumers "wrongly" bought product 1 minus the fraction of consumers "wrongly" bought product 2. When $\alpha=0$ (no homophily), (14) boils down to (3) in Section 3. The next three lemmas present useful properties regarding the steady-state market shares for a fixed $\widehat{x}$.

Lemma 4 *Fix $\psi\in[1/2,1)$. (i) There is a unique ψ_L satisfying (11) and a unique ψ_R satisfying (12), with $0<\psi_R<\psi<\psi_L<1$. (ii) Both ψ_L and ψ_R are increasing in ψ; ψ_L is increasing in α and ψ_R is decreasing in α. (iii) $\psi_R\geq 1-\psi_L$.*

Observe that ψ_R is the probability that a type R consumer "wrongly" buy product 1, and $1-\psi_L$ is the probability that a type L consumer "wrongly" buy product 2. As the degree of homophily α increases, the probability that each type of consumers are informed of wrong products only decreases, and thus both ψ_R and $1-\psi_L$ decrease (as shown in part (ii) of Lemma 4). The reason for $\psi_R\geq 1-\psi_L$ (as in part (iii)) is that the market share of firm 1, ψ, is weakly larger than 1/2. Due to the component of random connection, it means that the probability that a type L consumer is aware of product 1 is weakly higher than the probability that a type R consumer is aware of product 2. As a result, compared to a type L consumer, a type R consumer is more likely to buy the wrong product.

Lemma 5 *Fix* $\widehat{x} \in [1/2, 1)$. *(i) There is a unique steady-state* ψ *(and* ψ_L *and* ψ_R *as well), which satisfies* $\psi \in [\widehat{x}, 1)$. *(ii)* $\psi_R < \psi < \psi_L$ *and* ψ *strictly increases in* $\widehat{x}$.

By Lemma 5, a given full-information market share $\widehat{x}$ induces a unique steady-state market share ψ. The next lemma shows how network connectivity and the degree of homophily affect ψ.

Lemma 6 *Fix* $\widehat{x} \in [1/2, 1)$. *(i) Under the single-friend network or the infinite-friend network, or any other generic network but with* $\alpha = 1$, $\psi = \widehat{x}$. *(ii) Suppose* $\widehat{x} \in (1/2, 1)$. *Under any generic network with* $\alpha < 1$, $\psi > \widehat{x}$ *and* ψ *decreases in* α.

The most revealing result in Lemma 6 is that, under generic networks, firm 1's steady-state market share ψ decreases in the degree of homophily α. To understand the intuition, let us consider two polar cases: random connection and extreme homophily ($\alpha = 1$). Under random connection, each consumer is more likely to be aware of product 1 than product 2 due to the nature of random connection and the fact that $\widehat{x} > 1/2$. Then there are more R type consumers wrongly buying product 1 than L type consumers wrongly buying product 2, resulting in ψ being bigger than $\widehat{x}$. However, in the case of extreme homophily, there is no consumers buying wrong products, because every friend of a L (R) type consumer is of L (R) type, and thus every consumer is aware of the right product in steady state. As a result, the steady-state market share $\psi = \widehat{x}$. The general case ($\alpha \in (0, 1)$) is just a combination of the above two polar cases. As α increases, each type of consumers become more likely to be aware of their preferred products, and thus there are fewer consumers buying wrong products. This means that $\widehat{x}$ ($> 1/2$) translates into a smaller additional market share for firm 1 among partially informed consumers, i.e., ψ decreases.

Figure 6 illustrates the relationship between α and the shape of the $\psi(\widehat{x})$ curve. In the figure, $\lambda = 1/4$, $p_1 = p_2 = p_3 = 1/3$ in network 1 and $p_2 = p_3 = p_4 = p_5 = 1/4$ in network 2. Under both networks, the $\psi(\widehat{x})$ curve becomes more straight as α increases.

Similar to the basic model of random connections, we need to characterize the curvature of $\psi(\widehat{x})$. Let $z_L \equiv 1 - (1-\alpha)\psi - \alpha\psi_L$ (and $z_R \equiv (1-\alpha)\psi + \alpha\psi_R$) be the probability that a type L (R) consumer receives "wrong" information from a friend.

Lemma 7 *(i) Under any generic network* $\{p_k\}$, $\frac{d^2\psi}{d\widehat{x}^2}|_{\widehat{x}=1/2} = 0$ *and* $\lim_{\lambda\to 1} \frac{d^2\psi}{d\widehat{x}^2} = 0$ *for any* $\alpha \in [0, 1]$, *and* $\lim_{\alpha\to 1} \frac{d^2\psi}{d\widehat{x}^2} = 0$. *(ii) For any* $\alpha \in [0, 1]$, *under well-connected networks (*$p_k = 0$ *for all* $k \leq \overline{k}$ *and* $\overline{k}$ *is large),* $\frac{d^2\psi}{d\widehat{x}^2} \to 0$, *and under the single-friend network,* $\frac{d^2\psi}{d\widehat{x}^2} = 0$. *(iii) Under any generic network, if* λ *is big enough or if* α *is small enough, then* $\frac{d^2\psi}{d\widehat{x}^2} \geq 0$ *when* $\widehat{x} \leq 1/2$ *and* $\frac{d^2\psi}{d\widehat{x}^2} \leq 0$ *when* $\widehat{x} \geq 1/2$.

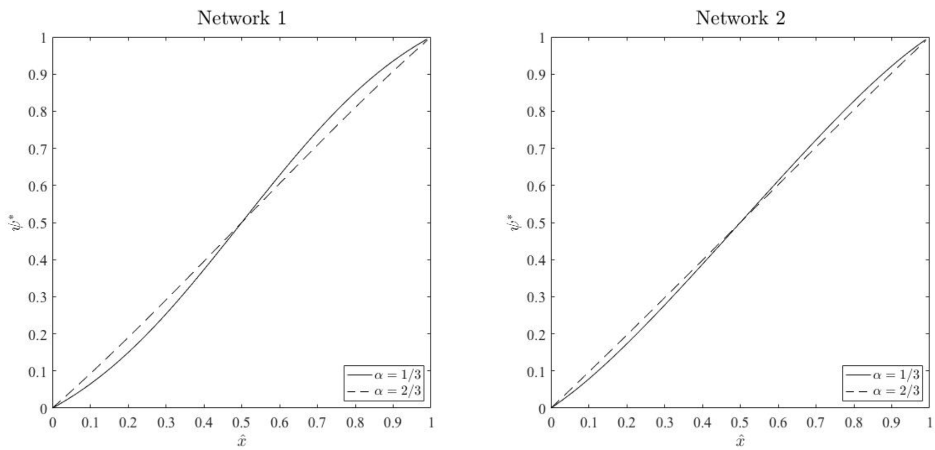


Figure 6: The $\psi(\widehat{x})$ Curve as α Changes

Lemma 7 specifies a set of sufficient conditions for $\psi(\widehat{x})$ to be convex when $\widehat{x} \leq 1/2$ and concave when $\widehat{x} \geq 1/2$.[26] In the remaining of this section we will focus on the set of $\{p_k\}$, λ and α such that $\psi(\widehat{x})$ satisfies the above convexity/concavity property.

Next we endogenize $\widehat{x}$ and characterize steady-state equilibria. The pricing equations (first-order conditions) have the same form as in the case of random connections:

$$\begin{aligned} P_1 &= \frac{2t\psi}{\frac{d\psi}{d\widehat{x}}}, P_2 = \frac{2t(1-\psi)}{\frac{d\psi}{d\widehat{x}}}, \\ \widehat{x} &= \frac{1}{2} + \frac{\Delta}{2t} - \frac{2\psi - 1}{\frac{d\psi}{d\widehat{x}}}, \end{aligned} \tag{15}$$

where by (11), (12) and (13),

$$\begin{aligned} \frac{d\psi}{d\widehat{x}} &= \frac{\psi_L - \psi_R}{1 - [\widehat{x}\frac{\partial \psi_L}{\partial \psi} + (1-\widehat{x})\frac{\partial \psi_R}{\partial \psi}]} \\ &= \frac{1 - (1-\lambda)[\sum_k p_k(z_L^k + z_R^k)]}{1 - (1-\lambda)(1-\alpha)[\widehat{x}\frac{\sum_k kp_k z_L^{k-1}}{1-(1-\lambda)\alpha\sum_k kp_k z_L^{k-1}} + (1-\widehat{x})\frac{\sum_k kp_k z_R^{k-1}}{1-(1-\lambda)\alpha\sum_k kp_k z_R^{k-1}}]}. \end{aligned} \tag{16}$$

By Lemma 5, $\frac{d\psi}{d\widehat{x}}$ is strictly positive. Equations (15) and (11)-(13) jointly determine the equilibrium pair of $(\widehat{x}, \psi)$, which we denote as $(\widehat{x}_e, \psi_e)$.

Proposition 6 *There is a unique candidate equilibrium (satisfying the steady-state equations and the first-order conditions), which satisfies* $\widehat{x}_e \in [\frac{1}{2}, \frac{1}{2} + \frac{\Delta}{2t}]$, *and if* $\Delta = 0$ *then it is symmetric:*

[26] For part (iii) to hold, λ could be relatively small and α could be of intermediate values. For instance, in the two examples of Figure 6, $\lambda = 1/4$ and $\alpha = \frac{1}{3}$ or $\frac{2}{3}$, but $\psi(\widehat{x})$ always has the desired convexity/concavity property.

$\widehat{x}_e = \psi_e = 1/2$. *The candidate equilibrium is an equilibrium under either of the following conditions: (i)* $\{p_k\}$ *is the single-friend network or a well-connected network; (ii)* λ *or* α *is relatively large.*

Similar to Proposition 1, conditions (i) and (ii) specified in Proposition 6 are sufficient conditions under which the second-order conditions are satisfied globally. Intuitively, under either condition in the limit the model converges to the Hotelling benchmark. Again, these two conditions are far from being necessary. Even when λ is relatively small and α is relatively small so that the second-order conditions are not satisfied globally, each firm's profit function could still be single-peaked, meaning that the candidate equilibrium satisfying the first-order conditions is indeed the equilibrium.[27]

4.1 Symmetric firms

With symmetric firms ($\Delta = 0$), Proposition 6 shows that the unique equilibrium is symmetric, with $\psi_e = \widehat{x}_e = 1/2$, and $P_1^e = P_2^e \equiv P^e = t/(\frac{d\psi}{d\widehat{x}}|_{\widehat{x}=1/2})$. Moreover, by (11) and (12), $\psi_L = 1 - \psi_R$, and $z_L = z_R$. Then (16) can be simplified as:

$$\frac{d\psi}{d\widehat{x}}|_{\widehat{x}=1/2} = \frac{(1-2\psi_R)\{1-(1-\lambda)\alpha \sum_k kp_k[(1-\alpha)/2+\alpha\psi_R]^{k-1}\}}{1-(1-\lambda)\sum_k kp_k[(1-\alpha)/2+\alpha\psi_R]^{k-1}}. \tag{17}$$

The next proposition characterizes the impact of homophily on the equilibrium price.

Proposition 7 *Suppose the two firms are symmetric. (i) If* $\{p_k\}$ *is the single-friend network or the infinite-friend network, then* $P^e = t$ *for any* α*. (ii) Under any other generic network, the equilibrium price* $P_e < t$ *(unless* $\alpha = 1$*) and is monotonically increasing in the degree of homophily* α*.*

Proposition 7 shows that homophily softens competition and increases the equilibrium price.[28] This result differs significantly from the one in Campbell (2019), where the degree of homophily does not affect the equilibrium price. The underlying reason for the difference is again the dynamic learning effect, which is absent in Campbell's (2019) model. The intuition for our result is already indicated in Lemma 6. As the degree of homophily α increases, each type of consumers (L or R) is more likely to be informed of their preferred products, and overall there are fewer consumers buying wrong products. This means that, if a firm cuts its

[27] With $\Delta = 0$, in each of the four examples in Figure 6, each firm's profit function is single-peaked in its own price when the other firm's price is fixed at its candidate equilibrium price.

[28] The impact of homophily is not restricted to market competition. Galeotti and Mattozi (2011) study how homophily among voters affects politicians' choice of policy platforms. Golub and Jackson (2012) show that homophily slows down the speed of social learning in a population initially endowed with heterogeneous opinions. Campbell et al. (2019) find that homophily in the social media network will increase political polarization.

price and thus expands its full-information market share $\widehat{x}$ above $1/2$, it can only induce fewer partially informed consumers to wrongly buy its product. As a result, the dynamic learning effect is dampened, and the steady-state market share becomes closer and less sensitive to the full-information market share $\widehat{x}$, which softens competition and increases the price.

Next we examine the impact of homophily on welfare. Similar to earlier analysis, total welfare W equals the gains from trade minus the total transportation costs incurred. Specifically,

$$W = V - 2[\frac{t}{4}\frac{1}{2}\psi_L + \frac{3t}{4}\frac{1}{2}\psi_R] = V - \frac{t}{4} - \frac{t}{2}\psi_R. \tag{18}$$

The second equality follows from the fact that $\psi_L = 1 - \psi_R$. In the expression of (18), $t/4$ is the total transportation costs if all consumers buy right products, ψ_R is the fraction of consumers who buy "wrong" products, and on average each of these consumers suffers an additional transportation costs of $t/2$. Similarly, consumer surplus CS can be written as

$$CS = W - P^e = V - \frac{t}{4} - \frac{t}{2}\psi_R - t/(\frac{d\psi}{d\widehat{x}}|_{\widehat{x}=1/2}). \tag{19}$$

Proposition 8 *Suppose the two firms are symmetric. (i) Total welfare W is increasing in α. (ii) Under the single-friend network or the infinite-friend network, consumer surplus CS is increasing in α. (iii) Under any other generic network $\{p_k\}$, if $p_1 \leq p_2$, then CS is decreasing in α for $\alpha \leq 1/2$; if $p_1 = 0$, then CS is decreasing in α for $\alpha \leq 3/4$.*

The result that homophily improves total welfare is intuitive. As the degree of homophily increases, consumers are more likely to be informed of their preferred products and thus fewer consumers buy wrong products, which decreases the total transportation costs and increases total welfare. As to consumer surplus, besides the information effect mentioned above, which always benefits consumers, there is a pricing effect which works in the opposite direction. In particular, homophily increases the price and thus hurts consumers. Part (iii) of Proposition 8 shows that the overall effect of homophily on consumer welfare is negative if α is not too large. The pricing effect tends to dominate, because an increase in α only prevents an additional fraction of consumers from buying wrong products, but the resulting increase in price hurts all consumers. Notice that this result is qualitatively different from Campbell (2019), in which homophily always improves consumer welfare. This is because the pricing effect of homophily is absent in his model.

In fact, numerical examples suggest that the overall effect of homophily on consumer welfare could be negative for the entire domain of α. Using the same networks as before with $t = 1$ and $\lambda = 1/4$, Figure 7 plots how the equilibrium price, total welfare, and consumer surplus change as α varies. Under both networks, consumer surplus is monotonically decreasing in α for any α.

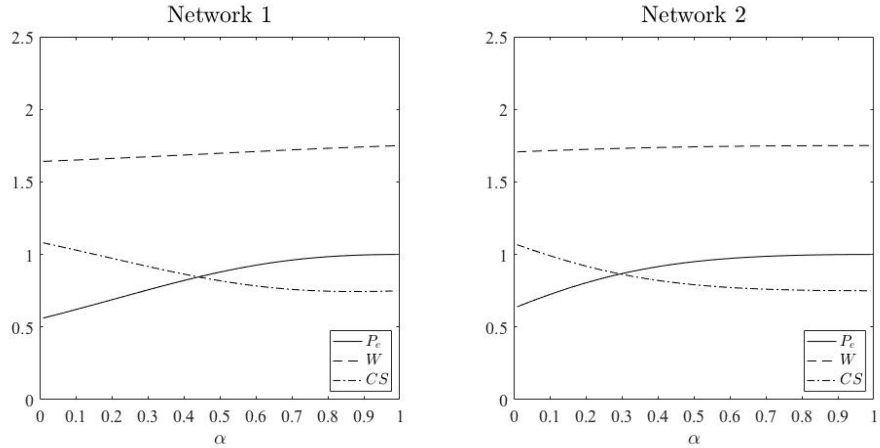


Figure 7: Homophily and Welfare with Symmetric Firms

4.2 Asymmetric firms

In this subsection, we consider asymmetric firms with $0 < \Delta \le t$, and focus on how the degree of homophily affects the equilibrium prices, market shares and welfare.

Proposition 9 *(i) Under the single-friend network or the infinite-friend network, the equilibrium prices and market shares are the same as those in the standard Hotelling model for any α. (ii) Consider other generic networks, and suppose that Δ is not too large so that ψ_e is not too far away from 1/2. The equilibrium market share of firm 1, ψ_e, is decreasing in α. Moreover, if $\alpha < 1$, then $\frac{d\psi}{d\widehat{x}}|_{\widehat{x}_e} > 1$, $\psi_e > \widehat{x}^H = \frac{1}{2} + \frac{\Delta}{6t}$, and $P_2^e < P_2^H = t - \frac{\Delta}{3}$.*

The most illuminating result in Proposition 9 is that, when the two firms are not too asymmetric, an increase in the degree of homophily reduces the equilibrium market share of the advantaged firm. In some sense, it implies that homophily dampens the advantage of the advantaged firm. The underlying intuition for this result is the same as that for part (ii) of Lemma 6: homophily dampens the dynamic learning effect. That is, as the degree of homophily α increases, there are fewer consumers buying wrong products, which means that the same $\widehat{x} > 1/2$ would translate into a smaller additional market share among partially informed consumers for firm 1.

What happens when Δ is relatively large? Numerical examples show that the result in part (ii) of Proposition 9 is still robust. Using the same networks and parameter values as before and with $\Delta = 2/3$, Figure 8 plots how the equilibrium market share and prices change with α. The figure illustrates that the result in part (ii) of Proposition 9 holds globally: for the whole domain of α, firm 1's equilibrium market share is decreasing in α. In addition, Figure 8 demonstrates that both firms' prices are increasing in α, the same pattern as in the case

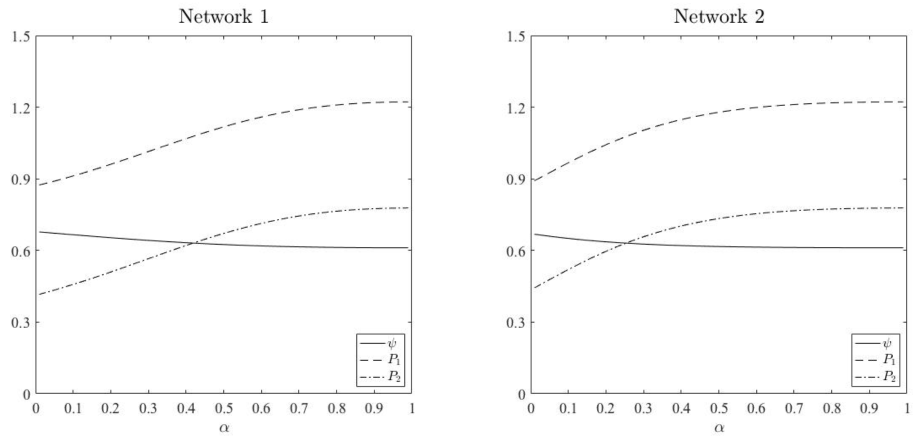


Figure 8: The Impact of Homophily on Prices and Market Share

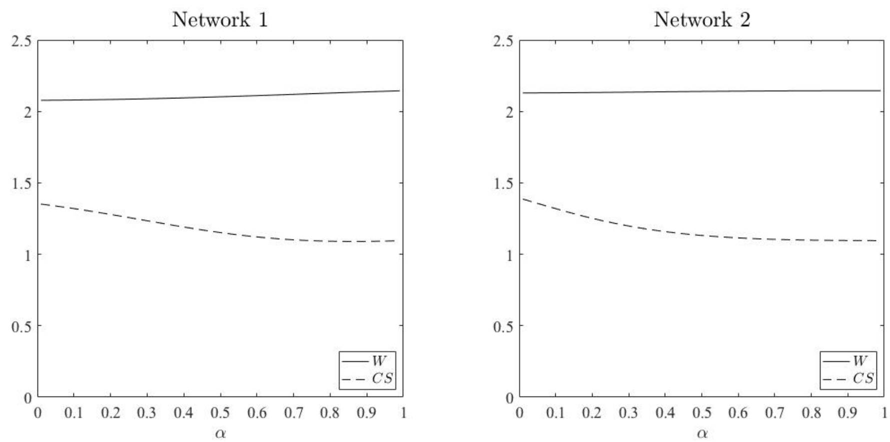


Figure 9: Homophily and Welfare with Asymmetric Firms

with symmetric firms. Overall, the conclusion is that an increase in the degree of homophily α softens competition and dampens firm 1's advantage.

Next we examine the impact of homophily on total welfare W and consumer surplus CS. In particular, it can be calculated that

$$\begin{aligned} W &= V + \Delta[\widehat{x}\psi_L + (1-\widehat{x})\psi_R] - \frac{t}{2} + t\widehat{x}(1-\widehat{x})(\psi_L - \psi_R), \\ CS &= W - P_1^e\psi_e - P_2^e(1-\psi_e). \end{aligned}$$

With the same parameter values as in Figure 8, Figure 9 plots W and CS as α changes. The same pattern emerges under both networks: as α increases, total welfare W increases but consumer surplus CS decreases. Note that this pattern is the same as the one with symmetric firms, for the same underlying reason. As α increases, the competition between firms is softened and prices increase. The negative impact of this pricing effect dominates the positive impact of the information effect, overall making consumers worse off.

5 Conclusion and Discussion

We study a dynamic model of price competition with differentiated products. Each generation of consumers learns about available products from their friends of the previous generation. The social network, which links consumers across generations, affects the evolution of consumers' awareness of products and firms' long-term market shares. Focusing on steady-state equilibria, we examine how the structure of the social network influences market shares, prices, and welfare. In our model, due to the dynamic learning process, firms also compete for partially informed consumers in future periods. As a result, competition is more intense compared to the Hotelling benchmark.

In the basic model with random connections, we find that the intensity of competition is non-monotonic in network connectivity. In particular, under relatively well connected networks, a further increase in network connectivity softens competition. As a result, while total welfare is increasing in network connectivity, consumer surplus is non-monotonic since the impact of network connectivity on equilibrium price is not monotonic. With asymmetric firms, the advantage of the advantaged firm is amplified by the dynamic learning process, leading to a bigger market share for the advantaged firm and a smaller market share and a lower price for the other firm relative to the Hotelling benchmark. In an extension with homophily, we find that an increase in the degree of homophily dampens the advantage of the advantaged firm and softens competition. Again, consumer welfare is non-monotonic in the degree of homophily.

We have assumed that prices remain fixed once they are set in the beginning. Allowing firms to adjust prices across periods would not change our qualitative results, as long as firms are patient enough so that they care about steady-state profits only. To see this, note that at the steady-state equilibrium neither firm has an incentive to unilaterally change its price. This is because this requirement has been incorporated in the equilibrium condition: given the other firm's price, a firm's price maximizes its steady-state profit. Of course, when firms are not very patient so that they also care about their profits in the path leading to the steady-state, then firms might adjust prices across periods. This setting is hard to analyze since prices may not settle in equilibrium, and we leave it for future research. However, if there are non-negligible costs for price adjustments, then firms will also behave according to the steady-state equilibrium.

In our model, the friends of each new generation of consumers are of the previous generation only. At the expense of technical complications, we can extend the model to settings in which the friends of each new generation are from previous generations. For instance, generation T

consumers could be friends with the $T-1$ and $T-2$ generations. However, extending our model to these settings will not qualitatively change our results, because firms are still competing for partially informed consumers in future periods due to the dynamic learning process. In these new settings, compared to our basic model, the steady state will be reached more slowly, since market shares in earlier periods will have more persistent impact on the demand in later periods. As a result, we conjecture that the steady-state demand will be more sensitive to the full-information market share and prices, which leads to more intense competition.

Appendix

Proof of Lemma 1.

Proof. We first show that, given $\widehat{x}$, there is a steady-state ψ, at which the $H(\psi)$ curve crosses the 45-degree line. By (3), it is clear that $H(\psi)$ is continuous in ψ. Moreover,

$$\lim_{\psi \to 0} H(\psi) = \lambda \widehat{x} > 0, \ \ \lim_{\psi \to 1} H(\psi) = 1 - \lambda(1 - \widehat{x}) < 1.$$

Therefore, there exists a $\psi \in (0, 1)$ satisfying $\psi = H(\psi)$.

For the uniqueness of ψ, it suffices to show that $\frac{\partial^3 H}{\partial \psi^3} \geq 0$ for all ψ within $(0, 1)$, which implies that $H(\psi)$ crosses the 45-degree line at most once within domain $(0, 1)$. By (3),

$$\frac{\partial^3 H}{\partial \psi^3} \propto \sum_{k \geq 3} p_k k(k-1)(k-2)[(1-\widehat{x})\psi^{k-3} + \widehat{x}(1-\psi)^{k-3}] \geq 0.$$

Thus ψ is unique.

Next, based on (3),

$$H(\widehat{x}) = \widehat{x} + (1-\lambda)\widehat{x}(1-\widehat{x}) \sum_k p_k[\widehat{x}^{k-1} - (1-\widehat{x})^{k-1}] \geq \widehat{x}.$$

The inequality follows from the fact that $\widehat{x}^{k-1} - (1-\widehat{x})^{k-1} \geq 0$ when $\widehat{x} \geq 1/2$. Combining with the earlier result that $\lim_{\psi \to 1} H(\psi) < 1$, we conclude that the unique ψ must be within $[\widehat{x}, 1)$. Moreover, $H(\widehat{x}) \geq \widehat{x}$ means that the $H(\psi)$ curve crosses the 45-degree line from above. This property further implies that the steady-state ψ is globally stable, as $\psi_{T+1} = H(\psi_T) > \psi_T$ when $\psi_T < \psi$ and $\psi_{T+1} = H(\psi_T) < \psi_T$ for $\psi_T > \psi$.

Finally, to show the monotonicity of ψ in $\widehat{x}$, let $\widehat{x}_2 > \widehat{x}_1 \geq 1/2$, and ψ_j be the corresponding steady-state ψ with $\widehat{x}_j$. That is, $\psi_j = H(\widehat{x}_j, \psi_j)$. By (3), we have

$$\frac{\partial H}{\partial \widehat{x}} = 1 - (1-\lambda) \sum_k p_k[(\psi)^k + (1-\psi)^k] > 0,$$

since the term of summation is less than 1. This implies that $H(\widehat{x}_2, \psi_1) > H(\widehat{x}_1, \psi_1) = \psi_1$. Now consider the case with $\widehat{x}_2$. The fact that $H(\widehat{x}_2, \psi_1) > \psi_1$ implies that at ψ_1 the $H(\psi)$ curve lies above the 45-degree line. Because the $H(\psi)$ curve crosses the 45-degree line from above, we must have $\psi_2 > \psi_1$. ■

Proof of Lemma 2.

Proof. Based on (3),

$$H(\widehat{x}) = \widehat{x} + (1-\lambda)\widehat{x}(1-\widehat{x}) \sum_k p_k[\widehat{x}^{k-1} - (1-\widehat{x})^{k-1}]. \tag{20}$$

Part (i). Under the single-friend network, (20) becomes $H(\widehat{x}) = \widehat{x}$ with $p_1 = 1$, and thus $\psi = \widehat{x}$. Under the infinite-friend network, when $k \to \infty$, both $\sum_k p_k \widehat{x}^{k-1}$ and $\sum_k p_k(1-\widehat{x})^{k-1}$ go to 0. Therefore, again $H(\widehat{x}) = \widehat{x}$ and $\psi = \widehat{x}$.

Part (ii). Consider any generic network. In (20), $\sum_k p_k[\widehat{x}^{k-1} - (1-\widehat{x})^{k-1}] > 0$ when $\widehat{x} > 1/2$ and $p_k > 0$ for some finite $k \geq 2$, which implies that $H(\widehat{x}) > \widehat{x}$. It means that $\psi > \widehat{x}$, because ψ is unique given $\widehat{x}$ and the $H(\psi)$ curve crosses the 45-degree line from above.

Part (iii). Under the k-friend network, the steady-state equation (3) is written as

$$\psi = \widehat{x} + (1-\lambda)[(1-\widehat{x})(\psi)^k - \widehat{x}(1-\psi)^k] \equiv H(\psi, k). \tag{21}$$

Fixing $\widehat{x}$, denote ψ_k as the solution to (21). For $k \geq 2$, to show that $\psi_{k+1} < \psi_k$, it is sufficient that $H(\psi_k, k+1) < \psi_k$; that is, under the $(k+1)$-friend network, $H(\psi_k)$ lies below the 45-degree line (recall that at ψ_{k+1}, the $H(\psi, k+1)$ curve crosses the 45 degree line from above by Lemma 1).

$$\begin{aligned} H(\psi_k, k+1) &= \widehat{x} + (1-\lambda)[(1-\widehat{x})(\psi_k)^{k+1} - \widehat{x}(1-\psi_k)^{k+1}] \\ &= \widehat{x} + (1-\lambda)[(1-\widehat{x})(\psi_k)^{k} - \widehat{x}(1-\psi_k)^{k}] \\ &\quad -(1-\lambda)\psi_k(1-\psi_k)[(1-\widehat{x})(\psi_k)^{k-1} - \widehat{x}(1-\psi_k)^{k-1}] \\ &< \widehat{x} + (1-\lambda)[(1-\widehat{x})(\psi_k)^{k} - \widehat{x}(1-\psi_k)^{k}] = H(\psi_k, k) = \psi_k. \end{aligned}$$

The inequality holds for $k \geq 2$ because $(1-\widehat{x})(\psi_k)^{k-1} - \widehat{x}(1-\psi_k)^{k-1} > 0$ with $\psi_k > \widehat{x} > 1/2$ based on part (ii). ■

Proof of Lemma 3.

Proof. Part (i). We can calculate the second-order derivative as follows:

$$\frac{d^2\psi}{d\widehat{x}^2} = \frac{d\psi}{d\widehat{x}}(1-\lambda)\frac{\frac{d\psi}{d\widehat{x}}\sum_k k(k-1)p_k[(1-\widehat{x})\psi^{k-2} - \widehat{x}(1-\psi)^{k-2}] - 2\sum_k kp_k[\psi^{k-1} - (1-\psi)^{k-1}]}{1-(1-\lambda)\sum_k kp_k[(1-\widehat{x})\psi^{k-1} + \widehat{x}(1-\psi)^{k-1}]}. \tag{22}$$

When $\widehat{x} = 1/2$, $\psi = 1/2$. It can be readily verified that the numerator of the fraction in the RHS of (22) is 0. Therefore, $\frac{d^2\psi}{d\widehat{x}^2}|_{\widehat{x}=1/2} = 0$.

Part (ii). Under well-connected networks, $\psi \simeq x$. And thus $\frac{d^2\psi}{d\widehat{x}^2} \simeq 0$.

For the rest of the proof, it is easier to work with $\widehat{x}$ as a function of ψ. By the steady-state equation (3), we have:

$$\widehat{x} = \frac{\psi - (1-\lambda)\sum p_k \psi^k}{1-(1-\lambda)\sum p_k[\psi^k + (1-\psi)^k]} \equiv \frac{A(\psi)}{A(\psi) + A(1-\psi)}, \tag{23}$$

where

$$A(\psi) = \psi - (1-\lambda)\sum p_k\psi^k, \text{and}$$
$$A(1-\psi) = 1 - \psi - (1-\lambda)\sum p_k(1-\psi)^k.$$

In order to show that $\psi(\widehat{x})$ is convex (concave) in $\widehat{x}$ when $\widehat{x} < 1/2$ ($\widehat{x} > 1/2$), it is enough to show that the inverse function, $\widehat{x}(\psi)$, is concave (convex) in ψ when $\psi < 1/2$ ($\psi > 1/2$). Taking various derivatives, we get

$$\begin{aligned}
A'(\psi) &= 1-(1-\lambda)\sum p_k k\psi^{k-1}, \\
A'(1-\psi) &= -1+(1-\lambda)\sum p_k k(1-\psi)^{k-1}, \\
A''(\psi) &= -(1-\lambda)\sum_{k\geq 2} p_k k(k-1)\psi^{k-2} \leq 0, \\
A''(1-\psi) &= -(1-\lambda)\sum_{k\geq 2} p_k k(k-1)(1-\psi)^{k-2} \leq 0, \\
\frac{d\widehat{x}}{d\psi} &= \frac{A'(\psi)A(1-\psi)-A(\psi)A'(1-\psi)}{[A(\psi)+A(1-\psi)]^2} > 0, \text{ since } \frac{d\psi}{d\widehat{x}} > 0.
\end{aligned}$$

Note that $A'(\psi)A(1-\psi)-A(\psi)A'(1-\psi) > 0$ since $\frac{d\widehat{x}}{d\psi} > 0$. For the second-order derivative, we have

$$\begin{aligned}
\frac{d^2\widehat{x}}{d\psi^2} = \; & \frac{(1-\lambda)}{[A(\psi)+A(1-\psi)]^3}\{[A(\psi)+A(1-\psi)]\sum_{k\geq 2} p_k k(k-1)[(1-\psi)^{k-2}A(\psi)-\psi^{k-2}A(1-\psi)] \\
& +2[A'(\psi)A(1-\psi)-A(\psi)A'(1-\psi)]\sum p_k k[\psi^{k-1}-(1-\psi)^{k-1}]\}. \qquad (24)
\end{aligned}$$

Part (iii). For the single-friend network, from (24) it can be verified that $\frac{d^2\widehat{x}}{d\psi^2} = 0$ for any $\psi \in [0,1]$. For the two-friend network, by (24) we have

$$\frac{d^2\widehat{x}}{d\psi^2} \propto \{2\lambda[A(\psi)+A(1-\psi)]+4[A'(\psi)A(1-\psi)-A(\psi)A'(1-\psi)]\}(2\psi-1).$$

Since the term in the braces is strictly positive, $\frac{d^2\widehat{x}}{d\psi^2}$ has the same sign as $2\psi-1$, which is strictly positive when $\psi > 1/2$ and strictly negative when $\psi < 1/2$.

For the 3-friend network, by (24) we have

$$\frac{d^2\widehat{x}}{d\psi^2} \propto [A(\psi)+A(1-\psi)-3(1-\lambda)\psi(1-\psi)][1-(1-\lambda)\psi(1-\psi)](2\psi-1).$$

Define the term in the first bracket as $c_3(\psi)$, and it is enough to show that $c_3(\psi) > 0$. In particular,

$$\begin{aligned}
c_3(\psi) &= [1-(1-\lambda)(\psi^3+(1-\psi)^3)-3(1-\lambda)\psi(1-\psi)] \\
&= \lambda\left[\psi^3+(1-\psi)^3+3\psi(1-\psi)\right] > 0,
\end{aligned}$$

where the equality uses the fact that $\psi^3+(1-\psi)^3+3\psi(1-\psi) = 1$.

For any sparsely-linked network, the relevant terms are just combinations of those under the k-friend networks with $k \leq 3$. Therefore, the result also holds.

Part (iv). For any generic network, by (24), we have $\lim_{\lambda\to 1} \frac{d^2\widehat{x}}{d\psi^2} = 0$ for any ψ.

Now consider k-friend networks with $k \geq 4$, under which

$$\begin{aligned}\frac{d^2\widehat{x}}{d\psi^2} \propto\ & -(k-1)[(1-\psi)\psi^{k-2} - \psi(1-\psi)^{k-2} + (1-\lambda)\psi^{k-2}(1-\psi)^{k-2}(2\psi-1)] \\ & \times \frac{A(\psi)+A(1-\psi)}{[A(\psi)+A(1-\psi)] - k(1-\lambda)[\psi^{k-1}A(1-\psi) + (1-\psi)^{k-1}A(\psi)]} + 2[\psi^{k-1} - (1-\psi)^{k-1}].\end{aligned}$$

Collecting the above terms without coefficient $(1-\lambda)$, we have

$$\psi^{k-2}[(k+1)\psi - (k-1)] + (1-\psi)^{k-2}[(k+1)\psi - 2] \equiv c_k(\psi).$$

For the 4-friend and 5-friend networks, we have, respectively,

$$c_4(\psi) = \psi^2(5\psi - 3) + (1-\psi)^2(5\psi - 2),$$

$$c_5(\psi) = \psi^3[6\psi - 4] + (1-\psi)^3[6\psi - 2],$$

which are positive if and only if $\psi \geq 1/2$ (the term is 0 when $\psi = 1/2$, and its derivative is positive). Therefore, for both networks, if λ is large enough, then $\frac{d^2\widehat{x}}{d\psi^2}$ is positive when $\psi > 1/2$ and negative when $\psi < 1/2$. For k-friend networks with k being large, we have $c_k(\psi) \to 0$ for any $\psi \in (0,1)$, or the result weakly holds.

To summarize, for any generic network, if λ is large enough, then $\frac{d^2\widehat{x}}{d\psi^2} \geq 0$ when $\psi > 1/2$ and $\frac{d^2\widehat{x}}{d\psi^2} \leq 0$ when $\psi < 1/2$. ■

Proof of Proposition 1.

Proof. We first show that a candidate equilibrium exists. With $\widehat{x}$ being the horizontal axis and ψ the vertical axis, the steady-state equation (3) defines a SS-curve and the pricing equation (6) defines a PE-curve. A candidate equilibrium is an intersection of these two curves. It is obvious that both curves are continuous. By (3), when $\widehat{x} = 1/2$, we have $\psi = 1/2$. Thus $(\frac{1}{2}, \frac{1}{2})$ is the starting point of the SS-curve. By (6), when $\widehat{x} = 1/2$, we have $\psi \geq 1/2$, because $\frac{d\psi}{d\widehat{x}} > 0$ by Lemma 1. Therefore, the starting point of the PE-curve is weakly above that of the SS-curve. Next consider $\widehat{x} = \frac{1}{2} + \frac{\Delta}{2t}$. By Lemma 2 on the SS-curve we have $\psi(\widehat{x} = \frac{1}{2} + \frac{\Delta}{2t}) \geq \widehat{x} = \frac{1}{2} + \frac{\Delta}{2t}$. By (6), $\psi(\widehat{x} = \frac{1}{2} + \frac{\Delta}{2t}) = 1/2$ on the PE-curve, because $\frac{d\psi}{d\widehat{x}} > 0$. Therefore, at $\widehat{x} = \frac{1}{2} + \frac{\Delta}{2t}$, the PE-curve is weakly below the SS-curve. By continuity, the two curves must intersect at some $\widehat{x} \in [\frac{1}{2}, \frac{1}{2} + \frac{\Delta}{2t}]$, which is a candidate equilibrium $\widehat{x}_e$. Since $\widehat{x}_e \in [\frac{1}{2}, \frac{1}{2} + \frac{\Delta}{2t}]$, we must have $\widehat{x}_e = 1/2$ when $\Delta = 0$, which implies that $\psi_e = 1/2$ as well.

Next we show the uniqueness of candidate equilibrium. The pricing equation (6) can be more compactly written as

$$\widehat{x} = \frac{1}{2} + \frac{\Delta}{2t} - (2\psi - 1)\frac{d\widehat{x}}{d\psi}. \tag{25}$$

The derivative of the RHS of (25) with respect to ψ equals to

$$-[(2\psi-1)\frac{d^2\widehat{x}}{d\psi^2}+2\frac{d\widehat{x}}{d\psi}],$$

which is strictly negative. To see this, by Lemma 3, $\frac{d^2\widehat{x}}{d\psi^2}\leq 0$ when $\psi\leq 1/2$, and $\frac{d^2\widehat{x}}{d\psi^2}\geq 0$ when $\psi\geq 1/2$. Therefore, $(2\psi-1)\frac{\partial^2\widehat{x}}{\partial\psi^2}\geq 0$ for any $\psi\in[0,1]$. Together with the fact that $\frac{d\widehat{x}}{d\psi}>0$, we have the desired result. In addition, the partial derivative of the RHS of (6) with respect to $\widehat{x}$ is negative. This means that the PE-curve is downward sloping. Note that the SS-curve is upward sloping because $\frac{d\psi}{d\widehat{x}}>0$. Thus, the two curves can have only one intersection; that is, the candidate equilibrium is unique.

Finally, we show the sufficiency of the first-order conditions by checking the second-order conditions. We will only prove the result for firm 1, as firm 2's situation is similar. In particular,

$$\frac{\partial^2\pi_1}{\partial P_1^2}\propto -2\frac{d\psi}{d\widehat{x}}+\frac{P_1}{2t}\frac{d^2\psi}{d\widehat{x}^2}.$$

Since $\frac{d\psi}{d\widehat{x}}>0$, and by Lemma 3, $\frac{d^2\psi}{d\widehat{x}^2}\leq 0$ when $\psi\geq 1/2$, we have $\frac{\partial^2\pi_1}{\partial P_1^2}<0$ when $\psi\geq 1/2$. When $\psi\leq 1/2$ (P_1 is relatively large), since $\frac{d^2\psi}{d\widehat{x}^2}\geq 0$ by Lemma 3, the sign of $\frac{\partial^2\pi_1}{\partial P_1^2}$ is indeterminate. To ensure $\frac{\partial^2\pi_1}{\partial P_1^2}\leq 0$, $|\frac{d^2\psi}{d\widehat{x}^2}|$ has to be small enough. By Lemma 3, $\lim_{\lambda\to 1}\frac{d^2\psi}{d\widehat{x}^2}=0$ and $\frac{d^2\psi}{d\widehat{x}^2}\to 0$ under either well-connected networks or the single-friend network. Therefore, the second-order condition $\frac{\partial^2\pi_1}{\partial P_1^2}\leq 0$ is satisfied if either λ is large enough or the network is either well-connected or the single-friend one. ■

Proof of Proposition 2.

Proof. Part (i). By (7), $\frac{d\psi}{d\widehat{x}}=1$ under both the single-friend network and the infinite-friend network. Then $P^e=t$ immediately follows. Now consider any generic network, with $p_l>0$ for some finite $l\geq 2$. Under these networks, it is obvious that

$$\sum_k kp_k(\frac{1}{2})^{k-1}>\sum_k p_k(\frac{1}{2})^{k-1}.$$

By (7), this implies that $\frac{d\psi}{d\widehat{x}}|_{\widehat{x}=1/2}>1$. Therefore, $P^e<t$.

Part (ii). Under the k-friend networks, by (7) we have

$$[\frac{d\psi}{d\widehat{x}}(k+1)-\frac{d\psi}{d\widehat{x}}(k)]|_{\widehat{x}=1/2}\propto(2-k)(\frac{1}{2})^k-(1-\lambda)(\frac{1}{2})^{2k-1},$$

which is positive if $k=1$, but is negative for any $k\geq 2$. The statement in the proposition immediately follows. ■

Proof of Proposition 3.

Proof. Part (i). It is sufficient to show that $[\frac{d\psi}{d\widehat{x}}(p_k') - \frac{d\psi}{d\widehat{x}}(p_k'')]|_{\widehat{x}=1/2} > 0$. By (7), $[\frac{d\psi}{d\widehat{x}}(p_k') - \frac{d\psi}{d\widehat{x}}(p_k'')]|_{\widehat{x}=1/2}$ has the same sign as

$$\sum_k (k-1)(p_k' - p_k'')(\frac{1}{2})^{k-1} + (1-\lambda)[\sum_k p_k'(\frac{1}{2})^{k-1}\sum_k kp_k''(\frac{1}{2})^{k-1} - \sum_k p_k''(\frac{1}{2})^{k-1}\sum_k kp_k'(\frac{1}{2})^{k-1}].$$

Since $(\frac{1}{2})^{k-1}$ is decreasing in k and $\{p_k''\}$ FOSD $\{p_k'\}$, $A \equiv \sum_k p_k''(\frac{1}{2})^{k-1} < \sum_k p_k'(\frac{1}{2})^{k-1} \equiv B$. The term $(k-1)(\frac{1}{2})^{k-1}$ is constant when k changes from 2 to 3, and decreases in k for $k \geq 3$. Since $\{p_k''\}$ FOSD $\{p_k'\}$ and $p_1'' = p_1'$, relative to $\{p_k'\}$, $\{p_k''\}$ puts higher probabilities on $k \geq 3$. Therefore, $Z \equiv \sum_k (k-1)(p_k' - p_k'')(\frac{1}{2})^{k-1} > 0$. Similarly, $C \equiv \sum_k kp_k''(\frac{1}{2})^{k-1} < \sum_k kp_k'(\frac{1}{2})^{k-1} \equiv D$, because the term $k(\frac{1}{2})^{k-1}$ is constant when k changes from 1 to 2, and is decreasing in k for $k \geq 2$. Note that A, B, C, and D are all smaller than 1. Moreover,

$$(A+D) - (B+C) = \sum_k (k-1)(p_k' - p_k'')(\frac{1}{2})^{k-1} = Z > 0.$$

Using more compact notations, we have

$$[\frac{d\psi}{d\widehat{x}}(p_k') - \frac{d\psi}{d\widehat{x}}(p_k'')]|_{\widehat{x}=1/2} \propto Z + (1-\lambda)(BC - AD).$$

If $BC \geq AD$, then we get the desired result that $[\frac{d\psi}{d\widehat{x}}(p_k') - \frac{d\psi}{d\widehat{x}}(p_k'')]|_{\widehat{x}=1/2} > 0$. Next consider the case that $BC < AD$. In particular,

$$Z + (1-\lambda)(BC - AD) > Z + BC - AD > B(C + Z) - AD > 0.$$

The first inequality holds since $BC < AD$. The second inequality uses the fact that $B < 1$. The last inequality holds because $B + C + Z = A + D$, $A < B < D$, and $A < C + Z < D$. Therefore, again $[\frac{d\psi}{d\widehat{x}}(p_k') - \frac{d\psi}{d\widehat{x}}(p_k'')]|_{\widehat{x}=1/2} > 0$.

Part (ii). By (7),

$$\frac{\partial(\frac{d\psi}{d\widehat{x}}|_{\widehat{x}=1/2})}{\partial\lambda} \propto \sum_k (1-k)p_k(\frac{1}{2})^{k-1},$$

which is negative if there is a finite $k \geq 2$ such that $p_k > 0$. Therefore, the equilibrium price increases in λ. ■

Proof of Proposition 4.

Proof. Since $(\frac{1}{2})^{k+1}$ is decreasing k, a FOSD change in $\{p_k\}$ reduces $\sum_k p_k(\frac{1}{2})^{k+1}$. By (8), this implies that W increases.

Next consider consumer surplus CS. For the k-friend networks, we can compute the difference in CS as k increases by 1 based on equation (9):

$$CS(k) - CS(k+1) \propto (\frac{1}{2})^{k+2}[(4k-9) + 11(1-\lambda)(\frac{1}{2})^k - (1-\lambda)^2(\frac{1}{2})^{2k-1}]. \qquad (26)$$

It can be verified that (26) is negative when $k = 1$. Therefore, $CS(k = 2) > CS(k = 1)$. This proves part (i). For part (ii), it can be verified that (26) is positive when $k \geq 3$. Therefore, CS is decreasing in k when $k \geq 3$. Finally, for part (iii), when $k = 2$ the term in the bracket in (26) becomes

$$-1 + \frac{11}{4}(1 - \lambda) - \frac{(1 - \lambda)^2}{8},$$

which is positive if $\lambda \leq \sqrt{113} - 10 \simeq 0.63$ and negative otherwise. The result immediately follows. ■

Proof of Proposition 5.

Proof. Part (i). By Lemma 2, under either network, $\psi = \widehat{x}$, and thus $\frac{d\psi}{d\widehat{x}} = 1$ at any $\widehat{x}$. Then the pricing equation (6) becomes $\widehat{x}_e = \frac{1}{2} + \frac{\Delta}{2t} - (2\widehat{x}_e - 1)$, which yields $\widehat{x}_e = \frac{1}{2} + \frac{\Delta}{6t} = \psi_e$. By (4), we get the desired expressions for P_1^e and P_2^e, which are the same as those in the Hotelling benchmark.

Part (ii). We first prove $\frac{d\psi_e}{d\widehat{x}}|_{\widehat{x}_e} > 1$. Define $r \equiv \frac{1}{\frac{d\psi_e}{d\widehat{x}}|_{\widehat{x}_e}}$. It is enough to show that $r < 1$. Suppose to the contrary, $r \geq 1$. But by (5), $r \geq 1$ implies that $\sum_k p_k D_k \leq 0$, where

$$D_k \equiv k[(1 - \widehat{x}_e)(\psi_e)^{k-1} + \widehat{x}_e(1 - \psi_e)^{k-1}] - [(\psi_e)^k + (1 - \psi_e)^k].$$

Since $\widehat{x}_e < \psi_e$ and $\psi_e > 1/2$,

$$D_k > k(1 - \psi_e)\psi_e[(\psi_e)^{k-2} + (1 - \psi_e)^{k-2}] - [(\psi_e)^k + (1 - \psi_e)^k] \equiv d_k. \tag{27}$$

It can be verified that $d_1 = 0$; and given any $k \geq 2$, for d_k to be less than 0, it is necessary that $\psi_e > \frac{3}{4}$. That is, if $\psi_e \leq \frac{3}{4}$, then $d_k \geq 0$ for all k and thus $\sum_k p_k D_k > 0$. Therefore, $r \geq 1$ implies that $\psi_e > \frac{3}{4}$.

Now we are ready to derive a contradiction. Given that $r \geq 1$ and $\psi_e > \frac{3}{4} \geq \frac{1}{2} + \frac{\Delta}{4t}$ (since $\Delta \leq t$), by (6), we have $\widehat{x}_e < \frac{1}{2} + \frac{\Delta}{2t} - \frac{\Delta}{2t} = \frac{1}{2}$. But, by (3), $\widehat{x}_e < 1/2$ implies that $\psi_e < \frac{1}{2}$. This is a contradiction. Therefore, it must be the case that $r < 1$.

Next we show that $\psi_e > \frac{1}{2} + \frac{\Delta}{6t}$. Suppose $\psi_e \leq \frac{1}{2} + \frac{\Delta}{6t}$. Given that $r < 1$, by (6) we have $\widehat{x}_e > \frac{1}{2} + \frac{\Delta}{2t} - \frac{\Delta}{3t} = \frac{1}{2} + \frac{\Delta}{6t}$. That is, $\widehat{x}_e > \psi_e$. But this contradicts the result in Lemma 2 that $\widehat{x}_e < \psi_e$. Therefore, in equilibrium $\psi_e > \frac{1}{2} + \frac{\Delta}{6t}$ must hold.

Finally, recall that $P_2^e = \frac{2t(1-\psi_e)}{\frac{d\psi_e}{d\widehat{x}}|_{\widehat{x}_e}}$. Given that $\psi_e > \frac{1}{2} + \frac{\Delta}{6t}$ and $\frac{d\psi_e}{d\widehat{x}}|_{\widehat{x}_e} > 1$, $P_2^e < t - \frac{\Delta}{3}$.

Part (iii). Denote $\frac{d\widehat{x}_k}{d\psi}$ as the derivative under the k-friend network. We first show that $\frac{d\widehat{x}_k}{d\psi}$ is increasing in k when ψ is greater than but close to 1/2. By earlier results, we have

$$\frac{d\widehat{x}_k}{d\psi} = \frac{1}{1 - (1 - \lambda)[\psi^k + (1 - \psi)^k]} - (1-\lambda)\frac{k\psi(1 - \psi)[\psi^{k-2} + (1 - \psi)^{k-2} - (1 - \lambda)\psi^{k-2}(1 - \psi)^{k-2}]}{[1 - (1 - \lambda)[\psi^k + (1 - \psi)^k]]^2}.$$

Letting

$$R \equiv \frac{1-(1-\lambda)[\psi^{k+1}+(1-\psi)^{k+1}]}{1-(1-\lambda)[\psi^{k}+(1-\psi)^{k}]} \geq 1,$$

and taking the difference, we get

$$\begin{aligned}\frac{d\widehat{x}_{k+1}}{d\psi}-\frac{d\widehat{x}_{k}}{d\psi} \quad &\propto \quad kR[\psi^{k-2}+(1-\psi)^{k-2}-(1-\lambda)\psi^{k-2}(1-\psi)^{k-2}]-[\psi^{k-1}+(1-\psi)^{k-1}] \\ &\quad -\frac{k+1}{R}[\psi^{k-1}+(1-\psi)^{k-1}-(1-\lambda)\psi^{k-1}(1-\psi)^{k-1}].\end{aligned}$$

When $\psi = 1/2$ and $k = 2$, we have

$$\frac{d\widehat{x}_3}{d\psi}-\frac{d\widehat{x}_2}{d\psi} \propto \frac{1}{2}(1-\lambda) > 0.$$

When $\psi = 1/2$ and $k \geq 3$, we have

$$\begin{aligned}\frac{d\widehat{x}_{k+1}}{d\psi}-\frac{d\widehat{x}_{k}}{d\psi} \quad &\propto \quad 2k[1-(1-\lambda)(\frac{1}{2})^{k}]-1-(k+1)[1-(1-\lambda)(\frac{1}{2})^{k-1}] \\ &\geq \quad k-2 > 0.\end{aligned}$$

By continuity, we conclude that $\frac{d\widehat{x}_{k+1}}{d\psi}-\frac{d\widehat{x}_{k}}{d\psi} > 0$ when ψ is close to $1/2$.

Now consider $k' > k$, and index the equilibrium variables by subscripts k and k'. We want to show $\psi_{e.k'} < \psi_{e,k}$. Suppose the opposite, $\psi_{e.k'} \geq \psi_{e,k}$, holds. By part (iii) of Lemma 2, the inequality means that $\widehat{x}_{e,k'} > \widehat{x}_{e,k}$, since under the k'-friend network the same $\widehat{x}$ leads to a smaller ψ than under the k-friend network. Under both networks, in equilibrium we have

$$\widehat{x}_e = \frac{1}{2}+\frac{\Delta}{2}-(2\psi_e-1)\frac{d\widehat{x}}{d\psi}|_{\psi_e}. \tag{28}$$

Regarding the derivative, we have $\frac{d\widehat{x}_k}{d\psi}|_{\psi_{e,k}} < \frac{d\widehat{x}_{k'}}{d\psi}|_{\psi_{e,k}} \leq \frac{d\widehat{x}_{k'}}{d\psi}|_{\psi_{e,k'}}$. The first inequality holds because $\frac{d\widehat{x}_k}{d\psi}$ is increasing in k, and the second inequality holds because $\widehat{x}$ is convex in ψ when $\psi \geq 1/2$ and $\psi_{e,k'} \geq \psi_{e,k} \geq 1/2$. Therefore, we have

$$(2\psi_{e,k}-1)\frac{d\widehat{x}_k}{d\psi}|_{\psi_{e,k}} < (2\psi_{e,k'}-1)\frac{d\widehat{x}_{k'}}{d\psi}|_{\psi_{e,k'}}.$$

Thus the RHS of (28) is smaller under the k'-friend network than under the k-friend network. However, $\widehat{x}_{e,k'} > \widehat{x}_{e,k}$ means that the LHS of (28) is larger under the k'-friend network. This is a contradiction. ■

Proof of Lemma 4.

Proof. Parts (i) and (ii). The proof is similar to that of Lemma 1. Define the LHS of (11) and (12) as $H_L(\psi_L)$ and $H_R(\psi_R)$, respectively. Note that both $H_L(\cdot)$ and $H_R(\cdot)$ are continuous functions. Given ψ, the steady-state ψ_L satisfies $\psi_L = H_L(\psi_L)$ and ψ_R satisfies $\psi_R = H_R(\psi_R)$. We first show that for any given ψ, there is a unique $\psi_L \in (\psi, 1)$. It can be verified that $H_L(\psi) = 1 - (1-\lambda)\sum_k p_k(1-\psi)^k \geq 1 - (1-\lambda)(1-\psi) > \psi$, and $H_L(1) = 1 - (1-\lambda)\sum_k p_k[(1-\alpha)(1-\psi)]^k < 1$. Thus the continuity of $H_L(\cdot)$ implies the existence of a $\psi_L \in (\psi, 1)$ which satisfies $\psi_L = H_L(\psi_L)$. To show the uniqueness, it is sufficient that $\frac{\partial^3 H_L}{\partial \psi^3} > 0$, which implies that the H_L-curve crosses the 45 degree line at most once. It is straightforward to check that $\frac{\partial^3 H_L}{\partial \psi^3} > 0$ holds, thus we have the uniqueness of ψ_L.

Next we show that ψ_L is increasing in ψ. Notice that $H_L(\psi) > \psi$ also implies that the H_L-curve crosses the 45 degree line from above. As ψ increases, the H_L-curve shifts up, which means that ψ_L increases, or $\frac{\partial \psi_L}{\partial \psi} > 0$. To show that ψ_L is increasing in α, consider $\alpha'' > \alpha'$, and denote the corresponding steady-state ψ_L as ψ'_L and ψ''_L, respectively. Since $\psi'_L > \psi$ and $\alpha'' > \alpha'$, by (11) we have $\psi'_L = H_L(\psi'_L, \alpha') < H_L(\psi'_L, \alpha'')$. Given that the H_L-curve crosses the 45 degree line from above, $\psi''_L > \psi'_L$ must hold.

Following similar steps, we can show the results regarding ψ_R. First, $H_R(0) > 0$ and $H_R(\psi) = (1-\lambda)\sum_k p_k \psi^k < \psi$. Second, $\frac{\partial^3 H_R}{\partial \psi^3} > 0$. These properties ensure a unique $\psi_R \in (0, \psi)$ satisfying $\psi_R = H_R(\psi_R)$, and that the H_R-curve crosses the 45 degree line from above. As ψ increases, the H_R-curve shifts up, which means that ψ_R increases, or $\frac{\partial \psi_R}{\partial \psi} > 0$. Finally, consider $\alpha'' > \alpha'$. Since $\psi'_R < \psi$, by (12) $\alpha'' > \alpha'$ implies that $\psi'_R = H_R(\psi'_R, \alpha') > H_R(\psi'_R, \alpha'')$. Given that the H_R-curve crosses the 45 degree line from above, $\psi''_R < \psi'_R$ must hold.

Part (iii). Denote $\phi_L \equiv 1 - \psi_L$. By (11) and (12), we have

$$\phi_L = (1-\lambda)\sum_k p_k[\phi_L + (1-\alpha)(1-\psi-\phi_L)]^k, \tag{29}$$

$$\psi_R = (1-\lambda)\sum_k p_k[\psi_R + (1-\alpha)(\psi-\psi_R)]^k. \tag{30}$$

Define the RHS of (29) as $G_L(\phi_L)$, and thus $G_L(\phi_L) = \phi_L$. Recall that the RHS of (30) is $H_R(\psi_R)$. Since $\psi \geq 1/2$, comparing (29) and (30), we have $H_R(y) \geq G_L(y)$ for any $y \leq \psi$. Therefore, $\phi_L = G_L(\phi_L) \leq H_R(\phi_L)$. The fact that $\phi_L \leq H_R(\phi_L)$ implies that $\psi_R \geq \phi_L = 1 - \psi_L$, since the H_R-curve crosses the 45 degree line from above. ■

Proof of Lemma 5.

Proof. Part (i). Denote $\psi_L(\psi)$ and $\psi_R(\psi)$ as the functions of the steady-state ψ_L and ψ_R when ψ changes. Then $H(\psi_L(\psi), \psi_R(\psi)) \equiv \widehat{x}\psi_L(\psi) + (1-\widehat{x})\psi_R(\psi)$, and the steady-state ψ satisfies $\psi = H(\psi_L(\psi), \psi_R(\psi))$. For the existence of a steady state, it suffices to show that (a) $H(\psi_L(\psi), \psi_R(\psi))$ is continuous, (b) $\lim_{\psi\to 0} H(\psi_L(\psi), \psi_R(\psi)) \geq 0$, and (c) $\lim_{\psi\to 1} H(\psi_L(\psi), \psi_R(\psi)) \leq 1$. Note that (a) holds since both $\psi_L(\psi)$ and $\psi_R(\psi)$ are continuous, and both (b) and (c) are satisfied since, by (11) and (12), $\psi_L \in [0, 1]$ and $\psi_R \in [0, 1]$.

Therefore, the existence of a steady-state ψ is ensured. For the uniqueness of ψ, it is enough to show that (d) $\frac{d^3}{d\psi^3}H(\psi_L(\psi), \psi_R(\psi)) \geq 0$, which implies that $H(\psi_L(\psi), \psi_R(\psi))$ crosses the 45-degree line at most once. Actually, combined with (b) and (c), property (d) also implies that the $H(\psi)$ curve crosses the 45-degree line from above at the steady-state ψ.

To show property (d), it is enough to show that $\frac{\partial^3 \psi_L}{\partial \psi^3} \geq 0$ and $\frac{\partial^3 \psi_R}{\partial \psi^3} \geq 0$, because $\frac{d^3}{d\psi^3}H(\psi_L(\psi), \psi_R(\psi)) = \widehat{x}\frac{\partial^3 \psi_L}{\partial \psi^3} + (1-\widehat{x})\frac{\partial^3 \psi_R}{\partial \psi^3}$. For that purpose, define $f(z) \equiv \sum_k p_k z^k$, $z_L \equiv 1-(1-\alpha)\psi - \alpha\psi_L$, and $z_R \equiv (1-\alpha)\psi + \alpha\psi_R$. It can be readily verified that $\frac{\partial^n}{\partial z^n}f(z) \geq 0$ for all n. Moreover, $\frac{\partial z_L}{\partial \psi} = -(1-\alpha+\alpha\frac{\partial \psi_L}{\partial \psi})$ and $\frac{\partial z_R}{\partial \psi} = (1-\alpha+\alpha\frac{\partial \psi_R}{\partial \psi})$. Differentiating (11) with respect to ψ yields

$$\frac{\partial \psi_L}{\partial \psi} = \frac{(1-\lambda)(1-\alpha)f'(z_L)}{1-(1-\lambda)\alpha f'(z_L)}.$$

The numerator of the above expression is positive. By Lemma 4, $\frac{\partial \psi_L}{\partial \psi} > 0$. Thus the denominator, $1-(1-\lambda)\alpha f'(z_L)$, is also positive. That $\frac{\partial \psi_L}{\partial \psi} > 0$ also implies that $\frac{\partial z_L}{\partial \psi} < 0$. Differentiating (11) repeatedly, we get

$$\begin{aligned}\frac{\partial^2 \psi_L}{\partial \psi^2} &= \frac{-(1-\lambda)f''(z_L)(\frac{\partial z_L}{\partial \psi})^2}{1-(1-\lambda)\alpha f'(z_L)},\\ \frac{\partial^3 \psi_L}{\partial \psi^3} &= \frac{(1-\lambda)[-f'''(z_L)(\frac{\partial z_L}{\partial \psi})^3 + 3\alpha f''(z_L)\frac{\partial z_L}{\partial \psi}\frac{\partial^2 \psi_L}{\partial \psi^2}]}{1-(1-\lambda)\alpha f'(z_L)}.\end{aligned}$$

Note that $\frac{\partial^2 \psi_L}{\partial \psi^2} < 0$, since the numerator is negative. On the other hand, $\frac{\partial^3 \psi_L}{\partial \psi^3} > 0$, because the numerator is positive since $\frac{\partial z_L}{\partial \psi} < 0$ and $\frac{\partial^2 \psi_L}{\partial \psi^2} < 0$.

Similarly, regarding ψ_R we have

$$\begin{aligned}\frac{\partial \psi_R}{\partial \psi} &= \frac{(1-\lambda)(1-\alpha)f'(z_R)}{1-(1-\lambda)\alpha f'(z_R)} > 0,\\ \frac{\partial^2 \psi_R}{\partial \psi^2} &= \frac{(1-\lambda)f''(z_R)(\frac{\partial z_R}{\partial \psi})^2}{1-(1-\lambda)\alpha f'(z_R)},\\ \frac{\partial^3 \psi_R}{\partial \psi^3} &= \frac{(1-\lambda)[f'''(z_R)(\frac{\partial z_R}{\partial \psi})^3 + 3\alpha f''(z_R)\frac{\partial z_R}{\partial \psi}\frac{\partial^2 \psi_R}{\partial \psi^2}]}{1-(1-\lambda)\alpha f'(z_R)}.\end{aligned}$$

The fact that $\frac{\partial \psi_R}{\partial \psi} > 0$ implies that $1-(1-\lambda)\alpha f'(z_R) > 0$ and $\frac{\partial z_R}{\partial \psi} > 0$. Then it is straightforward to check that $\frac{\partial^2 \psi_R}{\partial \psi^2} > 0$ and $\frac{\partial^3 \psi_R}{\partial \psi^3} > 0$. This completes the proof that the steady-state ψ is unique.

Next, we show that $\psi \geq \widehat{x}$. By (13),

$$H(\psi) = \widehat{x} + [(1-\widehat{x})\psi_R(\psi) - \widehat{x}(1-\psi_L(\psi))].$$

By earlier results, it suffices to show $H(\widehat{x}) \geq \widehat{x}$, which is equivalent to $(1-\widehat{x})\psi_R(\widehat{x}) - \widehat{x}(1-\psi_L(\widehat{x})) \geq 0$. Somewhat abusing the notations, let $z_L = 1-(1-\alpha)\widehat{x} - \alpha\psi_L$ and $z_R =$

$(1-\alpha)\widehat{x}+\alpha\psi_R$. When $\psi=\widehat{x}$, (11) becomes $1-\psi_L=(1-\lambda)\sum_k p_k z_L^k$ and (12) becomes $\psi_R=(1-\lambda)\sum_k p_k z_R^k$. By Lemma 4, $1-\psi_L\le\psi_R$ when $\psi=\widehat{x}\ge 1/2$, thus we have $z_L\le z_R$.

Note that

$$(1-\widehat{x})z_R-\widehat{x}z_L=\alpha[(1-\widehat{x})\psi_R(\widehat{x})-\widehat{x}(1-\psi_L(\widehat{x}))].$$

Thus it suffices to show that $(1-\widehat{x})z_R-\widehat{x}z_L\ge 0$ or $\frac{1-\widehat{x}}{\widehat{x}}\frac{z_R}{z_L}-1\ge 0$. Again by (11) and (12),

$$(1-\widehat{x})z_R-\widehat{x}z_L=\alpha(1-\lambda)\sum_k p_k[(1-\widehat{x}){z_R}^k-\widehat{x}{z_L}^k],$$

which leads to

$$\frac{1-\widehat{x}}{\widehat{x}}\frac{z_R}{z_L}-1=\alpha(1-\lambda)\sum_k p_k z_L^{k-1}[\frac{1-\widehat{x}}{\widehat{x}}(\frac{z_R}{z_L})^k-1]. \tag{31}$$

Now suppose $\frac{1-\widehat{x}}{\widehat{x}}\frac{z_R}{z_L}-1<0$. Since $z_L\le z_R$, $\frac{1-\widehat{x}}{\widehat{x}}(\frac{z_R}{z_L})^k-1\ge\frac{1-\widehat{x}}{\widehat{x}}\frac{z_R}{z_L}-1$ for all k. Given that $\alpha(1-\lambda)z_L^{k-1}<1$ for all k and the LHS of (31) is strictly negative, for each k, the term in the RHS of (31) is either positive or strictly less negative than the LHS. Thus, (31) cannot hold with equality, leading to a contradiction. Therefore, we must have $\frac{1-\widehat{x}}{\widehat{x}}\frac{z_R}{z_L}-1\ge 0$.

Part (ii). The result that $\psi_R<\psi<\psi_L$ directly follows from Lemma 4. Finally, to show that ψ is increasing in $\widehat{x}$, consider any $\widehat{x}'>\widehat{x}''\ge 1/2$, and denote the corresponding steady-state ψ as ψ_S' and ψ_S'', respectively. Note that $H(\psi_S'';\widehat{x}'')=\widehat{x}''\psi_L(\psi_S'')+(1-\widehat{x}'')\psi_R(\psi_S'')$, and $\psi_L(\psi_S'')>\psi_R(\psi_S'')$ by Lemma 4. Therefore, $H(\psi_S'';\widehat{x}')=\widehat{x}'\psi_L(\psi_S'')+(1-\widehat{x}')\psi_R(\psi_S'')>H(\psi_S'';\widehat{x}'')=\psi_S''$. Combined with the fact that the $H(\psi)$ curve crosses the 45-degree line from above, this implies $\psi_S'>\psi_S''$. This proves that ψ strictly increases in $\widehat{x}$. ■

Proof of Lemma 6.

Proof. Part (i). First consider the single-friend network. With $p_1=1$, (14) becomes

$$\psi-\widehat{x}=\frac{(1-\lambda)(1-\alpha)}{1-(1-\lambda)\alpha}(\psi-\widehat{x}),$$

which implies that $\psi=\widehat{x}$.

Under the infinite-friend network, by (11) and (12), we have $\psi_L=1$ and $\psi_R=0$. Then $\psi=\widehat{x}$ based on (13). Finally, consider the case of any generic network with $\alpha=1$. By (11) and (12), again we have $\psi_L=1$ and $\psi_R=0$. Therefore, $\psi=\widehat{x}$.

Part (ii). To show $\psi>\widehat{x}$, we follow similar steps as in the proof of part (ii) of Lemma 5, where the weak inequality is proved. With a generic network and $\widehat{x}>1/2$, following part (iii) of Lemma 4, we can show that $\psi_R>1-\psi_L$ if $\psi>1/2$, which leads to $\frac{z_R}{z_L}>1$. Then in part (ii) of Lemma 5, $\frac{1-\widehat{x}}{\widehat{x}}(\frac{z_R}{z_L})^k-1>\frac{1-\widehat{x}}{\widehat{x}}\frac{z_R}{z_L}-1$ for all k. Now suppose $\frac{1-\widehat{x}}{\widehat{x}}\frac{z_R}{z_L}-1=0$. Then $\frac{1-\widehat{x}}{\widehat{x}}(\frac{z_R}{z_L})^k-1>0$ for all $k\ge 2$. Therefore, the RHS of (31) is strictly positive, contradicting the equality. Therefore, $H(\widehat{x})>\widehat{x}$ and $\psi>\widehat{x}$.

Finally, we show that ψ decreases in α. By (11) and (12),

$$\begin{aligned}\frac{\partial\psi_L}{\partial\alpha} &= -(1-\lambda)f'(z_L)[-(1-\alpha)\frac{\partial\psi}{\partial\alpha}+\psi-\psi_L-\alpha\frac{\partial\psi_L}{\partial\alpha}] \\ &= \frac{(1-\lambda)f'(z_L)}{1-\alpha(1-\lambda)f'(z_L)}[(1-\alpha)\frac{\partial\psi}{\partial\alpha}-\psi+\psi_L] \\ &= \frac{\partial\psi_L}{\partial\psi}\frac{\partial\psi}{\partial\alpha}+\frac{(1-\lambda)f'(z_L)}{1-\alpha(1-\lambda)f'(z_L)}(1-\widehat{x})(\psi_L-\psi_R).\end{aligned}$$

The last equality follows from $\psi_L-\psi=(1-\widehat{x})(\psi_L-\psi_R)$. Similarly,

$$\begin{aligned}\frac{\partial\psi_R}{\partial\alpha} &= (1-\lambda)f'(z_R)[(1-\alpha)\frac{\partial\psi}{\partial\alpha}-\psi+\psi_R+\alpha\frac{\partial\psi_R}{\partial\alpha}] \\ &= \frac{(1-\lambda)f'(z_R)}{1-\alpha(1-\lambda)f'(z_R)}[(1-\alpha)\frac{\partial\psi}{\partial\alpha}-\psi+\psi_R] \\ &= \frac{\partial\psi_R}{\partial\psi}\frac{\partial\psi}{\partial\alpha}-\frac{(1-\lambda)f'(z_R)}{1-\alpha(1-\lambda)f'(z_R)}\widehat{x}(\psi_L-\psi_R).\end{aligned}$$

The last equality follows from $\psi_R-\psi=-\widehat{x}(\psi_L-\psi_R)$. Combine the above results,

$$\begin{aligned}\frac{\partial\psi}{\partial\alpha} &= \widehat{x}\frac{\partial\psi_L}{\partial\alpha}+(1-\widehat{x})\frac{\partial\psi_R}{\partial\alpha} \\ &= [\widehat{x}\frac{\partial\psi_L}{\partial\psi}+(1-\widehat{x})\frac{\partial\psi_R}{\partial\psi}]\frac{\partial\psi}{\partial\alpha} \\ &\quad +\widehat{x}(1-\widehat{x})(1-\lambda)(\psi_L-\psi_R)[\frac{f'(z_L)}{1-\alpha(1-\lambda)f'(z_L)}-\frac{f'(z_R)}{1-\alpha(1-\lambda)f'(z_R)}] \\ &\propto \frac{\frac{f'(z_L)}{1-\alpha(1-\lambda)f'(z_L)}-\frac{f'(z_R)}{1-\alpha(1-\lambda)f'(z_R)}}{1-[\widehat{x}\frac{\partial\psi_L}{\partial\psi}+(1-\widehat{x})\frac{\partial\psi_R}{\partial\psi}]}\text{ , because }\psi_L>\psi_R \\ &\propto \frac{f'(z_L)}{1-\alpha(1-\lambda)f'(z_L^*)}-\frac{f'(z_R)}{1-\alpha(1-\lambda)f'(z_R^*)}\text{ , because }\frac{\partial\psi}{\partial\widehat{x}}>0 \\ &< 0.\end{aligned}$$

The last inequality holds because $f'(z)$ is increasing in z and $z_L<z_R$, as shown earlier. ■

Proof of Lemma 7.

Proof. Parts (i) and (ii). We can compute

$$\begin{aligned}\frac{d^2\psi}{d\widehat{x}^2} &= \frac{\psi_L-\psi_R}{[1-[\widehat{x}\frac{\partial\psi_L}{\partial\psi}+(1-\widehat{x})\frac{\partial\psi_R}{\partial\psi}]]^2}\{2(\frac{\partial\psi_L}{\partial\psi}-\frac{\partial\psi_R}{\partial\psi})+[\widehat{x}\frac{\partial^2\psi_L}{\partial\psi^2}+(1-\widehat{x})\frac{\partial^2\psi_R}{\partial\psi^2}]\frac{d\psi}{d\widehat{x}}\} \\ &\propto (1-\lambda)(1-\alpha)\{\frac{2f'(z_L)}{1-(1-\lambda)\alpha f'(z_L)}-\frac{2f'(z_R)}{1-(1-\lambda)\alpha f'(z_R)} \\ &\quad +(1-\alpha)[(1-\widehat{x})\frac{f''(z_R)}{[1-(1-\lambda)\alpha f'(z_R)]^3}-\widehat{x}\frac{f''(z_L)}{[1-(1-\lambda)\alpha f'(z_L)]^3}]\frac{d\psi}{d\widehat{x}}\}.\end{aligned}$$

When $\widehat{x} = 1/2$, $\psi = 1/2$, $\psi_R = 1 - \psi_L$, and $z_L = z_R$, and thus $\frac{d^2\psi}{d\widehat{x}^2}|_{\widehat{x}=1/2} = 0$. It is also easy to see that $\lim_{\lambda\to 1} \frac{d^2\psi}{d\widehat{x}^2} = 0$ for any $\alpha \in [0, 1]$, and that $\lim_{\alpha\to 1} \frac{d^2\psi}{d\widehat{x}^2} = 0$.

Under well-connected networks, $f'(z) \to 0$ and $f''(z) \to 0$ for any $z \in [0, 1)$. Therefore, $\frac{d^2\psi}{d\widehat{x}^2} \to 0$. Under the single-friend network, $\psi = \widehat{x}$. Therefore, again $\frac{d^2\psi}{d\widehat{x}^2} = 0$.

Part (iii). When $\alpha = 0$, we know from Lemma 3 that $\frac{d^2\psi}{d\widehat{x}^2} \geq 0$ when $\widehat{x} \leq 1/2$ and $\frac{d^2\psi}{d\widehat{x}^2} \leq 0$ when $\widehat{x} \geq 1/2$. By continuity, the property continues to hold when α is small.

Regarding the result when λ is large, it is sufficient to show that $\frac{d^2\widehat{x}}{d\psi^2} \geq 0$ when $\widehat{x} \geq 1/2$ and $\frac{d^2\widehat{x}}{d\psi^2} \leq 0$ when $\widehat{x} \leq 1/2$. By (13), $\widehat{x} = \frac{\psi - \psi_R}{\psi_L - \psi_R}$. Then

$$\frac{d^2\widehat{x}}{d\psi^2} = \frac{\begin{array}{c} 2(\frac{\partial\psi_L}{\partial\psi} - \frac{\partial\psi_R}{\partial\psi})[(\psi_L - \psi)\frac{\partial\psi_R}{\partial\psi} + (\psi - \psi_R)\frac{\partial\psi_L}{\partial\psi} - (\psi_L - \psi_R)] \\ -(\psi_L - \psi_R)[(\psi_L - \psi)\frac{\partial\psi_R^2}{\partial\psi^2} + (\psi - \psi_R)\frac{\partial^2\psi_L}{\partial\psi^2}] \end{array}}{(\psi_L - \psi_R)^3}. \tag{32}$$

Because $\psi_L > \psi_R$ according to Lemma 5, the sign of $\frac{d^2\widehat{x}}{d\psi^2}$ is the same as that of the numerator in (32).

We first consider k-friend networks ($k \geq 2$). We can calculate that

$$\frac{\partial\psi_L}{\partial\psi} = \frac{(1-\lambda)(1-\alpha)kz_L^{k-1}}{1-\alpha(1-\lambda)kz_L^{k-1}}, \frac{\partial\psi_R}{\partial\psi} = \frac{(1-\lambda)(1-\alpha)kz_R^{k-1}}{1-\alpha(1-\lambda)kz_R^{k-1}},$$

$$\begin{aligned} \frac{\partial^2\psi_L}{\partial\psi^2} &= \frac{-(1-\lambda)(1-\alpha)k(k-1)z_L^{k-2}[(1-\alpha) + \alpha\frac{\partial\psi_L}{\partial\psi}]}{[1-\alpha(1-\lambda)kz_L^{k-1}]^2}, \\ \frac{\partial^2\psi_R}{\partial\psi^2} &= \frac{(1-\lambda)(1-\alpha)k(k-1)z_R^{k-2}[(1-\alpha) + \alpha\frac{\partial\psi_R}{\partial\psi}]}{[1-\alpha(1-\lambda)kz_R^{k-1}]^2}. \end{aligned}$$

Using the above results, the numerator in (32) has the same sign as Ω, where

$$\begin{aligned} \Omega =& 2[(\psi_L - \psi_R) - (\psi_L - \psi)\frac{\partial\psi_R}{\partial\psi} - (\psi - \psi_R)\frac{\partial\psi_L}{\partial\psi}](z_R^{k-1} - z_L^{k-1}) \\ &+ (k-1)(\psi_L - \psi_R)[-(\psi_L - \psi)z_R^{k-2}\frac{1-\alpha(1-\lambda)kz_L^{k-1}}{1-\alpha(1-\lambda)kz_R^{k-1}}((1-\alpha) + \alpha\frac{\partial\psi_R}{\partial\psi}) \\ &+ (\psi - \psi_R)z_L^{k-2}\frac{1-\alpha(1-\lambda)kz_R^{k-1}}{1-\alpha(1-\lambda)kz_L^{k-1}}((1-\alpha) + \alpha\frac{\partial\psi_L}{\partial\psi})]. \end{aligned}$$

When $\lambda \to 1$, we have $\psi_L \to 1$, $\psi_R \to 0$, $z_L \to (1-\alpha)(1-\psi)$, $z_R \to (1-\alpha)\psi$, $\frac{\partial\psi_R}{\partial\psi} \to 0$ and $\frac{\partial\psi_L}{\partial\psi} \to 0$. Then

$$\Omega \to (1-\alpha)^{k-1}\left[2(\psi^{k-1} - (1-\psi)^{k-1}) + (k-1)(-(1-\psi)\psi^{k-2} + \psi(1-\psi)^{k-2})\right]. \tag{33}$$

When $k=2$ or $k=3$, by (33) $\Omega \propto 2\psi-1$. When $k=4$, $\Omega \propto (2\psi-1)(2\psi^2-\psi(1-\psi)+2(1-\psi)^2)$, and when $k=5$, $\Omega \propto (\psi^2-(1-\psi)^2)(2\psi-1)^2$. In each case, Ω is strictly negative when $\psi < 1/2$ and strictly positive when $\psi > 1/2$. Moreover, Ω goes to 0 when k is large. Therefore, the concavity/convexity property holds for k-friend networks. Since any generic network is a weighted combination of k-friend networks, we conclude that the concavity/convexity property holds for generic networks if λ is large enough. ∎

Proof of Proposition 6.

Proof. We first show the existence of candidate equilibrium for $\Delta \in [0,t]$. With $\widehat{x}$ being the horizontal axis and ψ the vertical axis, an equilibrium is an intersection of the SS-curve defined by the steady-state equations (11)-(13) and the PE-curve defined by the pricing equation (15). Both curves are continuous. We can verify that $(\frac{1}{2},\frac{1}{2})$ is the starting point of the SS-curve. For the PE-curve, based on (15) and the fact that $\frac{d\psi}{\partial\widehat{x}} > 0$, $\psi \geq 1/2$ when $\widehat{x} = 1/2$. Therefore, the starting point of the PE-curve is weakly above the SS-curve. Now consider the ending point at $\widehat{x} = \frac{1}{2} + \frac{\Delta}{2t}$. By lemma 5, on the SS-curve $\psi(\frac{1}{2}+\frac{\Delta}{2t}) \geq \frac{1}{2}+\frac{\Delta}{2t}$. In addition, by the pricing equation (15), on the PE-curve $\psi = \frac{1}{2}$ when $\widehat{x} = \frac{1}{2}+\frac{\Delta}{2t}$, since $\frac{d\psi}{d\widehat{x}} > 0$. Therefore, the PE-curve is weakly below the SS-curve at the ending point. By continuity, the two curves must intersect at some $\widehat{x} \in [\frac{1}{2}, \frac{1}{2}+\frac{\Delta}{2t}]$, which is a candidate equilibrium $\widehat{x}_e$. Since $\widehat{x}_e \in [\frac{1}{2}, \frac{1}{2}+\frac{\Delta}{2t}]$, we must have $\widehat{x}_e = 1/2$ when $\Delta = 0$, which implies that $\psi_e = 1/2$ as well.

Next, the uniqueness of candidate equilibrium follows from the same proof as in Proposition 1. That the PE-curve is downward sloping follows from the pricing equation (15) and Lemma 7. The SS-curve is upward sloping by Lemma 5. Thus, the two curves can only have one intersection and the candidate equilibrium is unique.

Finally, following an argument similar to the proof of Proposition 1, by Lemma 7 the second-order conditions are satisfied if either λ or α is large enough, or the network is well-connected or the single-friend network. Thus, the sufficiency of the first-order conditions is guaranteed. ∎

Proof of Proposition 7.

Proof. Part (i). Following part (i) of Lemma 6, under both networks $\psi = \widehat{x}$. And thus $\frac{d\psi}{d\widehat{x}} = 1$ at any $\widehat{x}$. The results immediately follow.

Part (ii). It is sufficient to show that $\frac{d\psi}{d\widehat{x}}|_{\widehat{x}=\frac{1}{2}}$ is decreasing in α. Let $z \equiv (1-\lambda)\sum_k kp_k[(1-\alpha)/2+\alpha\psi_R]^{k-1}$. Then (17) can be written compactly as

$$\frac{d\psi}{d\widehat{x}}|_{\widehat{x}=\frac{1}{2}} = (1-2\psi_R)\frac{1-\alpha z}{1-z}. \tag{34}$$

By definition,

$$\frac{\partial z}{\partial \alpha} = (1-\lambda)\sum_k k(k-1)p_k[(1-\alpha)/2+\alpha\psi_R]^{k-2}(\psi_R - \frac{1}{2} + \alpha\frac{\partial \psi_R}{\partial \alpha}) < 0,$$

since by Lemma 4, $\psi_R < \psi = \frac{1}{2}$ and $\frac{\partial \psi_R}{\partial \alpha} < 0$. Now differentiating (34) with respect to α, we get

$$\begin{aligned} \frac{\partial^2 \psi}{\partial \widehat{x} \partial \alpha}|_{\widehat{x}=\frac{1}{2}} &\propto -2\frac{\partial \psi_R}{\partial \alpha}(1-\alpha z)(1-z) + (1-2\psi_R)[-z(1-z) + (1-\alpha)\frac{\partial z}{\partial \alpha}] \\ &= -2z(\psi_R - \frac{1}{2})(1-z) + (1-2\psi_R)[-z(1-z) + (1-\alpha)\frac{\partial z}{\partial \alpha}] \\ &= (1-2\psi_R)(1-\alpha)\frac{\partial z}{\partial \alpha} < 0. \end{aligned}$$

In the first equality above we used the result that $\frac{\partial \psi_R}{\partial \alpha} = \frac{z(\psi_R - \frac{1}{2})}{1-\alpha z}$, which can be derived from (12). Therefore, $\frac{d\psi}{d\widehat{x}}|_{\widehat{x}=\frac{1}{2}}$ is decreasing in α. ■

Proof of Proposition 8.

Proof. Part (i). Because ψ_R strictly decreases in α (Lemma 4), by (18) W strictly increases in α.

Part (ii). By Proposition 7, under the extreme networks $\frac{d\psi}{d\widehat{x}} = 1$ for all α. The result immediately follows part (i).

Part (iii). Using the notations and results in the proof of Proposition 7, and let $y \equiv (1-\alpha)/2 + \alpha\psi_R$, from (19) we compute

$$\begin{aligned} \frac{\partial CS}{\partial \alpha} &\propto (\psi_R - \frac{1}{2})(1-\alpha z)\frac{\partial \psi_R}{\partial \alpha} + \frac{1-z}{\psi_R - \frac{1}{2}}\frac{\partial \psi_R}{\partial \alpha} + \frac{-z(1-z) + (1-\alpha)\frac{\partial z}{\partial \alpha}}{1-\alpha z} \\ &\propto (\psi_R - \frac{1}{2})^2(1-\alpha z)z + (1-\alpha)(1-\lambda)\sum_k p_k k(k-1)y^{k-2}(\psi_R - \frac{1}{2} + \alpha\frac{\partial \psi_R}{\partial \alpha}) \\ &\propto \frac{1}{2}(1-2\psi_R)(1-\alpha z)^2 z - (1-\alpha)(1-\lambda)\sum_k p_k k(k-1)y^{k-2}. \end{aligned} \tag{35}$$

In the derivation we used the fact that $\psi_R < 1/2$.

To determine the sign of (35), define z' as follows and recall z:

$$\begin{aligned} z' &\equiv (1-\lambda)\sum_k p_k k(k-1)y^{k-2} = (1-\lambda)[2p_2 + 6p_3 y + ...], \\ z &= (1-\lambda)[p_1 + 2p_2 y + 3p_3 y^2 + ...]. \end{aligned}$$

Note that $y < 1/2$ because $\psi_R < 1/2$. Suppose $p_2 \geq p_1$. Since $y < 1/2$, $\frac{z'-z}{1-\lambda} \geq 2p_2(1-y) - p_1 > p_2 - p_1 \geq 0$; that is, $z' \geq z$. Then (35) is less than $\frac{1}{2}z' - (1-\alpha)z'$, which is negative if $\alpha \leq 1/2$. Next consider the case that $p_1 = 0$. Now $y \leq 1/2$ implies that $z' \geq 2z$. Then (35) is less than $\frac{1}{2}z' - 2(1-\alpha)z'$, which is negative if $\alpha \leq 3/4$. ■

Proof of Proposition 9.

Proof. Part (i). By part (i) of Lemma 6, $\psi = \widehat{x}$. Thus $\frac{d\psi}{d\widehat{x}} = 1$ at any $\widehat{x}$. Therefore, the market share equation and the pricing equation are the same as those in the standard Hotelling model, which implies the same equilibrium market share and prices.

Part (ii). We first show that $\frac{d\psi}{d\widehat{x}}$ is decreasing in α for $\widehat{x}$ close to 1/2. By the steady-state equations, we compute the relevant derivatives:

$$\frac{\partial\psi}{\partial\alpha} = \frac{\frac{\widehat{x}(1-\widehat{x})}{1-\alpha}(\psi_L - \psi_R)(\frac{\partial\psi_L}{\partial\psi} - \frac{\partial\psi_R}{\partial\psi})}{1 - [\widehat{x}\frac{\partial\psi_L}{\partial\psi} + (1-\widehat{x})\frac{\partial\psi_R}{\partial\psi}]},$$

$$\begin{aligned}\frac{\partial^2\psi}{\partial\widehat{x}\partial\alpha} \propto\ & (1-2\widehat{x})(\psi_L - \psi_R)(\frac{\partial\psi_L}{\partial\psi} - \frac{\partial\psi_R}{\partial\psi}) + \widehat{x}(1-\widehat{x})\frac{d\psi}{d\widehat{x}} \times \\ & \{(\frac{\partial\psi_L}{\partial\psi} - \frac{\partial\psi_R}{\partial\psi})^2(1 + \frac{1 - [\widehat{x}\frac{\partial\psi_L}{\partial\widehat{x}} + (1-\widehat{x})\frac{\partial\psi_R}{\partial\widehat{x}}]}{1 - [\widehat{x}\frac{\partial\psi_L}{\partial\psi} + (1-\widehat{x})\frac{\partial\psi_R}{\partial\psi}]}) + (\psi_L - \psi_R)(\frac{\partial^2\psi_L}{\partial\psi^2} - \frac{\partial^2\psi_R}{\partial\psi^2}) \\ & + (\frac{\partial\psi_L}{\partial\psi} - \frac{\partial\psi_R}{\partial\psi})\frac{1 - [\widehat{x}\frac{\partial\psi_L}{\partial\widehat{x}} + (1-\widehat{x})\frac{\partial\psi_R}{\partial\widehat{x}}]}{1 - [\widehat{x}\frac{\partial\psi_L}{\partial\psi} + (1-\widehat{x})\frac{\partial\psi_R}{\partial\psi}]}[\widehat{x}\frac{\partial^2\psi_L}{\partial\psi^2} + (1-\widehat{x})\frac{\partial^2\psi_R}{\partial\psi^2}]\frac{d\psi}{d\widehat{x}}\}. \qquad (36)\end{aligned}$$

When $\widehat{x} = 1/2$, we have $\psi = 1/2$, $\psi_R = 1 - \psi_L$, $z_L = z_R$, and $\frac{\partial\psi_L}{\partial\psi} - \frac{\partial\psi_R}{\partial\psi} = 0$. Therefore, (36) becomes

$$\frac{\partial^2\psi}{\partial\widehat{x}\partial\alpha}|_{\widehat{x}=\frac{1}{2}} \propto (\frac{\partial^2\psi_L}{\partial\psi^2} - \frac{\partial^2\psi_R}{\partial\psi^2})|_{\widehat{x}=\frac{1}{2}} = -[\frac{(1-\lambda)f''(z_L)(\frac{\partial z_L}{\partial\psi})^2}{1-(1-\lambda)\alpha f'(z_L)} + \frac{(1-\lambda)f''(z_R)(\frac{\partial z_R}{\partial\psi})^2}{1-(1-\lambda)\alpha f'(z_R)}] < 0.$$

This means that $\frac{d\psi}{d\widehat{x}}$ is decreasing in α when $\widehat{x} = 1/2$. By continuity, the result also holds for $\widehat{x}$ greater than but not too far away from 1/2.

Next we prove that ψ_e is decreasing in α. Specifically, consider $\alpha' > \alpha$, and index the endogenous variables by subscripts α and α'. We want to show that $\psi_{e,\alpha'} < \psi_{e,\alpha}$. Suppose the opposite, $\psi_{e,\alpha'} \geq \psi_{e,\alpha}$, holds. By part (ii) of Lemma 6, this means that $\widehat{x}_{e,\alpha'} > \widehat{x}_{e,\alpha}$, because the same $\widehat{x}$ leads to a smaller ψ under a larger α. By equation (15), the following equation holds for both α and α' in equilibrium:

$$\widehat{x}_e = \frac{1}{2} + \frac{\Delta}{2} - \frac{(2\psi_e - 1)}{\frac{d\psi}{d\widehat{x}}|_{\widehat{x}_e}}. \qquad (37)$$

Note that

$$\frac{d\psi_\alpha}{d\widehat{x}}|_{\widehat{x}_{e,\alpha}} > \frac{d\psi_{\alpha'}}{d\widehat{x}}|_{\widehat{x}_{e,\alpha}} > \frac{d\psi_{\alpha'}}{d\widehat{x}}|_{\widehat{x}_{e,\alpha'}}.$$

The first inequality holds because $\frac{\partial^2\psi}{\partial\widehat{x}\partial\alpha} < 0$, and the second inequality holds because ψ is concave in $\widehat{x}$ when $\widehat{x} \geq 1/2$ and $\widehat{x}_{e,\alpha'} > \widehat{x}_{e,\alpha}$. Therefore, we have

$$\frac{(2\psi_{e,\alpha} - 1)}{\frac{d\psi_\alpha}{d\widehat{x}}|_{\widehat{x}_{e,\alpha}}} < \frac{(2\psi_{e,\alpha'} - 1)}{\frac{d\psi_{\alpha'}}{d\widehat{x}}|_{\widehat{x}_{e,\alpha'}}}.$$

Now compare equation (37) under α and α'. Relative to those under α, the LHS is bigger under α', but the RHS is smaller under α'. This contradicts the fact that equation (37) holds under both α and α'.

Finally, we show that $\frac{d\psi}{d\widehat{x}}|_{\widehat{x}_e} > 1$ when $\alpha < 1$. Since $\frac{\partial^2\psi}{\partial\widehat{x}\partial\alpha} < 0$, we have

$$\frac{d\psi_\alpha}{d\widehat{x}}|_{\widehat{x}_{e,\alpha}} > \frac{d\psi_{\alpha=1}}{d\widehat{x}}|_{\widehat{x}_{e,\alpha}} = 1,$$

where the equality follows from part (i). The rest of the results can be proved by the same argument as in the proof of part (ii) of Proposition 5. ■

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