Inter-Dealer Trades in OTC Markets – Who Buys and Who Sells?

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Abstract

Oftentimes, dealers in an OTC market may not be able to trade with one another whenever they desire to do so, just as investors find it necessary to incur time and effort to buy and sell assets not traded in a centralized exchange. Moreover, an individual dealer can obviously only carry limited quantities of the asset over time and the inventory capacities may certainly differ among dealers. In this environment, dealers trade among themselves, whenever the opportunities arise, to rebalance inventories for facilitating the sale and purchase of the asset to and from investors. Such inter-dealer trades naturally resemble a core-periphery trading network documented empirically. Dealers with a smaller capacity occupy peripheral positions and dealers with a larger capacity occupy central positions in the network. The smaller capacity peripheral dealers trade to provide immediacy for the larger capacity central dealers. The model’s implications on dealers’ markups are consistent with the available empirical evidence. There are also novel implications on how inter-dealer market trading volumes vary with the asset supply and the diversity of dealers.

Keywords: OTC Market, Inter-Dealer Trades, Trading Network
JEL classifications: D53, D85, G23

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1 Introduction

Many financial assets, including government and corporate bonds, asset-backed securities, and derivatives, are traded in over-the-counter (OTC) markets instead of in centralized exchanges. Two distinguishing features of OTC markets are that trades are almost always intermediated by dealers of various kinds and that the dealers do not just trade with investors but also among themselves. Indeed, inter-dealer trades can account for a significant fraction of the overall transactions for a given asset.

It has long been recognized, going back to Ho and Stoll (1983), that dealers may trade among themselves for inventory risk concerns. In these models, a risk-averse trader having a greater exposure to some risky assets sells a certain fraction of his holding to another risk-averse trader with a lesser initial exposure to the mutual benefits of both. The common understanding seems to be that trading for inventory concerns is inherently linked to risk aversion. But must inter-dealer trades motivated by the sharing of inventory risks necessarily arise from risk aversion?

In many OTC markets, a dealer cannot take up a certain buy order from an investor unless the dealer possesses a large enough inventory of the asset beforehand if the dealer is not able to acquire the requisite amount of it from other dealers or investors at a short notice. On the other hand, if dealers' inventory capacities are not unbounded and if they cannot sell to others at will, a dealer can only meet a sell order from an investor if the dealer has sufficient spare inventory capacity at the given moment. Given such constraints, dealers may find it beneficial to trade with one another, whenever they are able to do so, to reach their optimal inventory levels to best prepare themselves for trading with investors. Such are inter-dealer trades in an OTC market motivated by the sharing of inventory risk, broadly understood, but not out of any consideration related to risk aversion.

In this paper, we extend the seminal random search models of the OTC market of Duffie, Gärleanu and Pedersen (2005) and Lagos and Rocheteau (2009) to study how dealers trade with one another for managing inventory levels for their future trading needs with investors. The point of departure is that, in our model, (1) dealers have only imperfect access to trading with other dealers and (2) they are heterogeneous in their inventory capacities. These are arguably very plausible assumptions. First, to be sure, in reality, there is not a frictionless platform on which dealers can continuously trade among themselves in a typical OTC market, just as investors must expend time and effort in buying and selling the asset. The heterogeneity in inventory capacity among dealers can result from differences in financing costs – dealers who finance asset purchases out of retained earnings and owners' equities can face different opportunity costs of funds, whereas dealers who finance asset purchases by borrowing can be charged different risk premia. The heterogeneity can also be due to risk management considerations or portfolio choices.

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1 As an example, the gross market value of global OTC derivatives totaled 38,286 billion US dollars in 2014 and 29,992 billion US dollars in 2015 (Bank for International Settlement, Semiannual OTC derivatives statistics, updated on May 4, 2016).

2 Li and Schürhoff (2014) show that in the period covered by their data set, 16 million out of 60 million transactions in municipal bonds are inter-dealer trades. A similar percentage of inter-dealer trade is also documented in Hollifield, Neklyudov and Spatt (2016).

3 In this paper, we do not attempt to model how the heterogeneity arises endogenously but instead restrict
With imperfect access to inter-dealer trading, it becomes imperative for dealers to choose the appropriate levels of inventory holding to be able to meet the uncertain future buy and sell orders from investors. In our model, dealers possessing different inventory capacities attain their respective optimal inventory holdings by buying and selling among themselves when the opportunities come, and that dealers of different inventory capacities play different roles in the inter-dealer market.

In particular, in our model, there is a given measure of what we call small dealers, each endowed with one unit of inventory capacity, and a given measure of what we call large dealers, each endowed with two units of inventory capacity. At the beginning of each period, investors who value the asset highly but have no asset in hand (high-valuation non-owners hereinafter) and investors who own a unit of the asset but do not value it (low-valuation owners hereinafter) enter the market to buy from and sell to dealers. Investors and dealers randomly meet in this investor-dealer market in which a given dealer can only sell to (buy from) an investor if the dealer has at least one unit of inventory (spare capacity) beforehand. The investor and the dealer in a transaction agree to a price reached via Nash bargaining. All investors who are on the market but fail to trade remain in the market. Once the investor-dealer trades are completed, and only then, a perfectly competitive inter-dealer market opens, through which dealers can rebalance their inventory holdings. Finally, at the end of the period, a fraction of high-valuation owners suffer exogenous liquidity shocks and turn into low-valuation owners, who then enter the investor-dealer market in the next period to sell their assets.

We restrict attention to studying steady-state equilibrium for brevity. We are able to obtain a multitude of analytical results from an apparently very complicated model, in which dealers of the two inventory capacities make decisions for every possible inventory level – decisions which affect and are affected by the steady-state distribution of agents.

Underlying most of the results to follow is a particular ranking of the marginal benefits of inventory, whereby the first unit of inventory is valued higher by a large dealer than by a small dealer, whereas the last unit of spare inventory capacity is valued higher by a large dealer than by a small dealer. The ranking is due to how the small, but not the large dealer, would exhaust his entire inventory capacity in acquiring one unit of the asset and how the large, but not the small dealer, need not fill up his entire inventory to already possess a unit of the asset for sale to investors.

The ranking implies that in equilibrium, depending on the asset supply and the extent of dealers’ heterogeneity, small dealers either always sell or always buy in the inter-dealer market. Large dealers, on the other hand, sell as well as buy in any equilibrium. If all small dealers only sell or only buy, they do not trade among themselves but only with large dealers. The large dealers, given that they sell as well as buy, trade among themselves, in addition to trading with small dealers. Altogether then, the trading patterns resemble a core-periphery trading network, as documented in Li and Schürrhoff (2014) and Hollifield, Neklyudov and Spatt (2016), in which large dealers are in the center, trading among themselves as well as with small dealers in the periphery, who do not trade with one another. Furthermore, the large dealers in the center always hold (weakly) more inventories than small dealers in the periphery do, given the former incur a lower opportunity cost in utilizing their first unit of inventory capacity than the
latter do in using up their only unit of inventory capacity, an implication also consistent with the findings in Li and Schürhoff (2014). Moreover, given the uniqueness of equilibrium, the direction of trade between small and large dealers is persistent for a given set of underlying parameters, another implication of the model that has empirical support with the findings in Li and Schürhoff (2014).

If the large core dealers are more interconnected and possess a greater inventory capacity, perhaps it seems intuitive that they should trade to provide inventory in particular and immediacy in general for the small peripheral dealers. It turns out that the exact opposite holds in our model – it is the small peripheral dealers who provide immediacy for the large core dealers in general, selling to the latter when inventory is in greatest demand but buying from them when liquidity is in shortest supply. The implication is due to the aforementioned ranking of the marginal benefits of inventory among different types of dealers. When the asset supply is at a relatively low level, in equilibrium, the inter-dealer market allocates inventory to the large core dealers with an empty inventory, who value inventory the most, through the sales of the asset by the small peripheral dealers to the large core dealers. When the asset supply is at a relatively high level, the inter-dealer market allocates spare capacity to the large core dealers starting out with a full inventory, who value spare capacity the most, through the purchases of the asset by the small peripheral dealers from the large core dealers.

The equilibrium in our model is constrained efficient in that the allocation of inventories and spare capacities among dealers falling out from inter-dealer trades in equilibrium coincide with the planning optimum. We show that rather intuitively, the optimum allocations serve to enable investors to buy and sell the asset most rapidly. The equilibrium allocations, on the other hand, are such that inventories and spare capacities are held by dealers who value them the most for their trading needs with investors. That the two allocations coincide perhaps is not surprising but more interestingly, it suggests that for efficiency, small peripheral dealers indeed should trade to provide immediacy for the large core dealers.

In addition to implications on the structure of trading relationships, our model yields a rich set of other testable implications for future empirical research on the OTC market. We investigate how market-tightness, inter-dealer trading price and volume change with respect to the asset supply and the diversity of dealers. Perhaps somewhat unexpected a priori is that the inter-dealer trading volume is “M-shaped” in response to changes in the asset supply – trading is most active when the asset supply is at a moderately low, but not the lowest, level and at a moderately high, but not the highest, level. Dealers trade among themselves to rebalance inventory, to which the need is greatest when either they find it hardest to acquire inventory or liquidity from investors, i.e., when the asset supply is at the lowest or the highest level. But precisely when the asset supply is at the lowest or the highest level, dealers who possess inventory (spare capacity) to sell (buy) can only be few and far between. In equilibrium, prices must then rise (fall) to dampen the demand (supply). In this way, trading is most active when the demand for and the supply of inventory are both at relatively high levels, arising from there being a moderately high or low asset supply. The inter-dealer trading volume is also non-monotonic with respect to the fraction of large dealers in the dealer population, reaching the maximum level when the fraction is at some intermediate level, whereby the dealer population is most diverse in terms of inventory capacity.
Related Literature

The main differences between this model and the seminal models of OTC markets in Duffie, Gärleanu and Pedersen (2005) and Lagos and Rocheteau (2009) are dealers’ imperfect access to the inter-dealer market and the heterogeneity of dealers’ inventory capacity – two features that make the present model more suitable for studying the inter-dealer trading relationship. In the two aforementioned papers, whenever a dealer trades with an investor, the dealer can instantaneously offset the transaction by trading in a perfectly competitive inter-dealer market that opens at all times. Such an environment, in which a dealer trades with another dealer only if and when he meets an investor, is arguably not the best environment to study inter-dealer trades as the trades have neither persistent direction nor particular structure. Moreover, dealers hold no inventory in this environment as long as they do not value the asset.

A host of recent papers are motivated to explain the empirical finding that the inter-dealer market exhibits a core-periphery structure. For instance, Neklyudov (2015) proposes a random search model assuming that dealers differ in their search abilities. He shows that the dealers with higher search abilities are more interconnected and hence are in the center. We highlight another factor, inventory capacity, that can influence a dealer’s position in inter-dealer trading relationships.

Another strand of investigation, a notable example of which is Wang (2017), models explicitly the formation of a core-periphery trading network in an environment in which a dealer can only trade with another dealer with which it has previously paid to set up an account. These models study how most dealers would choose to set up accounts with just a handful of dealers due to the usual network externality. In our model, the trading structure that emerges endogenously resembles a core-periphery trading network without any kind of network externality and where the establishment of a prior link is not needed for any two dealers to trade.

Ours is not the only model of an OTC market in which dealers hold inventory. Lagos, Rocheteau and Weill (2011) demonstrate that dealers hold inventory to speed up future trades when there is a negative shock knocking the market off the steady state, even if dealers do not value the asset. Weill (2011) shows that the same intuition also applies in a competitive dynamic market with a transient selling pressure. Dealers in our model hold inventory also to facilitate trade, as they do not have continuous access to the inter-dealer market. The difference is that they hold inventory even in the steady state and they gain by trading among themselves. Moreover, we can characterize the relationship between a dealer’s optimal inventory holding and the dealer’s inventory capacity.

Our paper departs from early models of inter-dealer trades motivated by inventory risk concerns that follow from Ho and Stoll (1983) and the modern variants, as in Atkeson, Eisfeldt and Weill (2015) and Üslü (2016), by assuming all traders are risk neutral. The dealers in our model are trading to mitigate inventory risks, as those in these models do, in that they trade to eliminate the risks of carrying an insufficient inventory or an insufficient inventory capacity as far as possible for their trading needs with investors.

Why dealers trade among themselves, other than for risk sharing, is a topic of active ongoing research. Colliard and Demange (2017) study post-issuance intermediation chains where each dealer has limited cash endowment, so that they need to trade with one another to amass the cash endowment of a group of dealers. Glode and Opp (2016) argue that when there is an adverse selection problem, a longer intermediation chain can narrow the information
gap between two successive dealers and help mitigate the problem of information asymmetry. The arguments in the two papers work, however, only when the order of transactions among dealers is exogenously fixed in a particular manner. Hugonnier, Lester and Weill (2016) and Shen, Wei and Yan (2016) assume that investors value the asset differently and show that those with intermediate valuations endogenously become dealers as they stay in the market to sell to investors with even higher valuations after buying. While their argument works in an environment where intermediaries value the asset, ours work even if dealers derive no flow payoff from holding the asset.

The rest of the paper is organized as follows. In Section 2, we set up the model and then study the model’s equilibrium. We discuss the model’s implications on dealers’ markups that follow from its predictions on the directions of trade between small and large dealers in Section 3 and compare those implications against the available empirical evidence. In Section 4, we discuss three extensions of the model and demonstrate how the major results hold in more general settings. We return to our basic model in Section 5 to study the model’s comparative statics. Section 6 concludes with discussions on the constrained efficiency of equilibrium in particular. All proofs are relegated to the Appendix, which also includes three respective Sections for the details of one extension of the model, two Propositions on the comparative statics of prices, and the formal analysis of the constrained efficiency of equilibrium.

2 Model and Analysis

2.1 Basic Environment

Time is discrete and runs forever. Two groups of agents – investors and dealers – buy and sell an asset with supply fixed at $A$ in an OTC market. A high-valuation investor derives a per period return of $\nu > 0$ in holding a unit of the asset, whereas low-valuation investors and dealers derive the same per period return normalized to zero. An individual investor can hold either zero or one unit of the asset at a time and can only buy or sell the asset through dealers of which there are two types: (1) small dealers, each of whom can hold up to one unit of the asset at a time and (2) large dealers, each of whom can hold up to two units. All agents are risk neutral and discount the future at the same factor $\beta$.

At the beginning of each period, a measure of $e$ investors enter the market as high-valuation investors with no assets in hand. Together with the entrants in previous periods who have yet to acquire a unit of the asset, they – the high-valuation non-owners – constitute the population of investor-buyers in the market. Among investors who do own a unit of the asset, the low-valuation owners become the investor-sellers in equilibrium.

Each period is divided into two subperiods. In the first subperiod, a decentralized investor-dealer market opens in which the bilateral meetings between investors and dealers take place. We assume that investor-buyers and dealers with at least a unit of the asset for sale (dealer-sellers hereinafter) meet in one segment of the market and investor-sellers and dealers with spare inventory capacity to buy (dealer-buyers hereinafter) meet in another segment of the market.

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4 A similar mechanism is proposed in Piazzesi and Schneider (2009) in their analysis of the housing market.
The matches in each market segment are formed in accordance with the same Mortensen-Pissarides constant-returns matching function, whereby, given market tightness $\theta \in [0, \infty)$ for the ratio of the measures of buyers to sellers for the given market segment, a seller meets a buyer at a probability $\eta(\theta) \in [0, 1]$, whereas a buyer meets a seller at the probability $\mu(\theta) = \eta(\theta) / \theta$. The meeting probability $\eta(\theta)$ satisfies the usual conditions:

$$\frac{\partial \eta}{\partial \theta} > 0; \quad \frac{\partial^2 \eta}{\partial \theta^2} < 0; \quad \lim_{\theta \to 0} \frac{\partial \eta}{\partial \theta} = 1; \quad \lim_{\theta \to \infty} \frac{\partial \eta}{\partial \theta} = 0.$$

With two market segments, there are two market tightness: (1) $\theta_{ID}$ for the ratio of the measures of investor-buyers to dealer-sellers and (2) $\theta_{DI}$ for the ratio of the measures of dealer-buyers to investor-sellers.

Prices in the investor-dealer market fall out of the bargaining between the buyers and sellers in the bilateral meetings in which the agents on the two sides are assumed to possess equal bargaining power. An individual dealer may search as both a dealer-buyer and a dealer-seller in the market in a given period but the meeting technology only allows the dealer to meet at most one investor-buyer and one investor-seller in the period. At the end of the subperiod, those high-valuation non-owners who succeed in buying a unit turn into new high-valuation owners, while those low-valuation owners who succeed in selling their units leave the market for good.

In the second subperiod, a competitive inter-dealer market opens, in which dealers buy and sell as many units of the asset among themselves as they see fit at a given market price, subject to their asset holdings and spare inventory capacities. Finally, at the end of the second subperiod, each high-valuation owner, except for those who have just purchased the asset in the current period, turns into a low-valuation owner at a probability $\delta \in (0, 1)$.

An apparent alternative to our assumed meeting technology is that investors and dealers search and match in one unified market in which an agent’s meeting probability depends on the ratio of all dealers to all investors. In this setup, there would be bilateral meetings between two sellers and between two buyers that cannot lead to any profitable exchanges between the agents concerned. Such no-trade meetings should not be common occurrences in reality. Hendershott and Madhavan (2015) report the increasing prevalence of electronic trading platforms for corporate bonds on which investors post their buy and sell orders. A dealer in any of these markets is then usually well informed of whether an investor is buying or selling before he initiates contact with the investor. Our assumed meeting technology embodies the standard assumption that how many matches of one type are formed depends only on the measures of agents who may become partners in such matches, whereas the unified market setup assumes that in addition the measures of other types of agents also matter and exert negative effects on the measures of the matches of the given type that would be formed, which means that for instance an investor-buyer’s matching probability decreases with the measures of

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5 The assumption of equal bargaining power is without loss of generality and merely serves to simplify.

6 This simplifying assumption can be understood as a discrete-time version of a continuous-time meeting process. If the time interval between two periods is small enough relative to the arrival rate of a meeting, then the probability of having more than one meeting per period approaches zero. The assumption can also be justified by a dealer’s limited execution capacity in reality. We should explain in Section 4 how the major results of the analysis survive while relaxing the restriction.
dealer-buyers and investor-sellers. Our assumed meeting technology has the virtue of choosing a simple versus a complicated setting when there is no compelling reason for choosing the latter.

A major difference between the present setting and that in the canonical models of Duffie, Garleanu and Pedersen (2005) and Lagos and Rocheteau (2009) is when and how often dealers have access to the competitive inter-dealer market. In the latter models, dealers can continuously access the competitive inter-dealer market. In such an environment, dealers do not hold any inventory at all in the steady state, as they can immediately offset any transaction with investors in the inter-dealer market. In contrast, the dealers in our model access the inter-dealer market only after or before they meet and trade with investors. Without continuous access to trading with other dealers, a dealer in our model can sell to an investor only if the dealer is holding at least a unit of the asset beforehand and thus the dealer may find it optimal not to load all units of the asset he acquires from investors at the first opportunity. In a similar vein, a dealer may find it optimal not to entirely fill up his inventory in the inter-dealer market, in anticipation of using the spare capacity for trading with an investor-seller if and when he meets one in the next period.

2.2 Value Functions

A small dealer, $S_i$, $i = 0, 1$, is either holding 0 or 1 unit of the asset at the beginning of a period when the investor-dealer market opens, whereas a large dealer, $L_i$, $i = 0, 1, 2$, may also be holding up to 2 units of the asset.

In the investor-dealer market, an investor-buyer meets a dealer-seller at probability $\mu(\theta_{ID})$. The dealer-seller can be an $S_1$, an $L_1$ or an $L_2$. Let $p_{IB,S_1}$, $p_{IB,L_1}$, and $p_{IB,L_2}$ be the respective prices at which the investor-buyer buys from these different dealers. Then, the investor-buyer has asset value,

$$U^B = \mu(\theta_{ID}) \left( \beta U^O_H - \frac{n^{SD}_1}{n^S_1} p_{IB,S_1} - \frac{n^{LD}_1}{n^S_1} p_{IB,L_1} - \frac{n^{LD}_2}{n^S_2} p_{IB,L_2} \right) + (1 - \mu(\theta_{ID})) \beta U^B, \quad (1)$$

where $n^{SD}_i$ and $n^{LD}_i$ are the respective measures of small and large dealers holding an $i$-unit inventory,

$$n^D_S = n^{SD}_1 + n^{LD}_1 + n^{LD}_2$$

the measure of all dealer-sellers, and $U^O_H$ the asset value of a high-valuation owner. In defining this value function, we assume that any meetings between an investor-buyer and a dealer-seller all yield a non-negative match surplus. Likewise, we shall assume that any meetings between an investor-seller and a dealer-buyer all yield a non-negative match surplus in defining the value functions in the following. In Lemma 1 below, we show that the assumptions are without loss of generality as they indeed hold in any equilibrium with active trading between investors and dealers.

A high-valuation owner derives a per period return $\upsilon$ from holding a unit of the asset and may turn into a low-valuation owner at probability $\delta$ at the end of the period. Hence,

$$U^O_H = \upsilon + \beta \left( \delta U^O_L + (1 - \delta) U^O_H \right), \quad (2)$$
where $U_{L}^{ON}$ denotes the asset value of a low-valuation owner who seeks to sell his unit of the asset. In each period, the investor-seller meets a dealer-buyer at probability $\eta(\theta_{DI})$. The dealer may be an $S_0$, an $L_0$ or an $L_1$. Let $p_{S_0,I_S}$, $p_{L_0,I_S}$, and $p_{L_1,I_S}$ be the respective prices at which the low-valuation investor sells to these different dealers. Hence,

$$U_{L}^{ON} = \eta(\theta_{DI}) \left( \frac{n_0^{SD}}{n_B} p_{S_0,I_S} + \frac{n_0^{LD}}{n_B} p_{L_0,I_S} + \frac{n_1^{LD}}{n_B} p_{L_1,I_S} \right) + (1 - \eta(\theta_{DI})) \beta U_{L}^{ON},$$

where

$$n_B^D = n_0^{SD} + n_0^{LD} + n_1^{LD}$$

is the measure of all dealer-buyers.

In addition to trading with investors in the investor-dealer market in the first subperiod, dealers may also trade among themselves in the second subperiod in the competitive inter-dealer market. Write $V_i^{SD}$ and $W_i^{SD}$, $i = 0, 1$, as the respective asset values of a small dealer entering the investor-dealer market in the first subperiod and the inter-dealer market in the second subperiod with an $i$-unit inventory. If the asset is traded in the inter-dealer market at price $p$,\n
$$W_0^{SD} = \max \{ \beta V_0^{SD}, \beta V_1^{SD} - p \},$$

$$W_1^{SD} = \max \{ p + \beta V_0^{SD}, \beta V_1^{SD} \},$$

$$V_0^{SD} = \mu(\theta_{DI}) (W_1^{SD} - p_{S_0,I_S}) + (1 - \mu(\theta_{DI})) W_0^{SD},$$

$$V_1^{SD} = \eta(\theta_{ID}) (p_{B_1,S_1} + W_0^{SD}) + (1 - \eta(\theta_{ID})) W_1^{SD}.$$ \hspace{1cm} (7)

In (4), an $S_0$ entering the inter-dealer market chooses between buying a unit in the market and not buying, whereas in (5), an $S_1$ chooses between selling the unit and not selling. Clearly, if the first dealer strictly prefers to buy where $p < \beta(V_1^{SD} - V_0^{SD})$, the second dealer must strictly prefer not to sell and vice versa. In (6), an $S_0$ entering the investor-dealer market meets an investor-seller at probability $\mu(\theta_{DI})$ and buys the unit from the investor at price $p_{S_0,I_S}$. In (7), an $S_1$ meets an investor-buyer at probability $\eta(\theta_{ID})$ and sells the unit to the investor at price $p_{B_1,S_1}$.

A large dealer can hold up to two units of the asset in inventory. The asset values, $V_i^{LD}$ and $W_i^{LD}$, $i = 0, 1, 2$, satisfy, respectively,

$$W_0^{LD} = \max \{ \beta V_0^{LD}, \beta V_1^{LD} - p, \beta V_2^{LD} - 2p \},$$

$$W_1^{LD} = \max \{ p + \beta V_0^{LD}, \beta V_1^{LD}, \beta V_2^{LD} - p \},$$

$$W_2^{LD} = \max \{ 2p + \beta V_0^{LD}, p + \beta V_1^{LD}, \beta V_2^{LD} \},$$

$$V_0^{LD} = \mu(\theta_{DI}) (W_1^{LD} - p_{L_0,I_S}) + (1 - \mu(\theta_{DI})) W_0^{LD},$$

$$V_1^{LD} = \eta(\theta_{IL}) (p_{B_1,S_1} + W_0^{LD}) + (1 - \eta(\theta_{IL})) W_1^{LD}.$$  \hspace{1cm} (11)

\footnote{In holding a unit in inventory and having one unit of spare inventory capacity, an $L_1$ is both a dealer-seller and a dealer-buyer in the given period.}
\[ V_1^{LD} = \mu (\theta_{DI}) (1 - \eta (\theta_{ID})) \left( W_2^{LD} - p_{L1,I_S} \right) + (1 - \mu (\theta_{DI})) \eta (\theta_{ID}) \left( p_{I_B,L1} + W_0^{LD} \right) + \mu (\theta_{DI}) \eta (\theta_{ID}) \left( p_{I_B,L1} - p_{L1,I_S} + W_1^{LD} \right) + (1 - \mu (\theta_{DI})) (1 - \eta (\theta_{ID})) W_1^{LD}, \]
\[ V_2^{LD} = \eta (\theta_{ID}) \left( p_{I_B,L2} + W_1^{LD} \right) + (1 - \eta (\theta_{ID})) W_2^{LD}. \]

2.3 Bargaining

Assuming equal bargaining power, the respective prices an investor-buyer pays to an \( S_1 \), an \( L_1 \), and an \( L_2 \) satisfy,
\[ \beta (U_H^{ON} - U_B) - p_{I_B,S1} = W_0^{SD} - W_1^{SD} + p_{I_B,S1}, \]
\[ \beta (U_H^{ON} - U_B) - p_{I_B,L1} = W_0^{LD} - W_1^{LD} + p_{I_B,L1}, \]
\[ \beta (U_H^{ON} - U_B) - p_{I_B,L2} = W_1^{LD} - W_2^{LD} + p_{I_B,L2}. \]

On the other hand, the respective prices an investor-seller receives from selling to an \( S_0 \), an \( L_0 \), and an \( L_1 \) satisfy,
\[ p_{S0,I_S} - \beta U_L^{ON} = W_1^{SD} - W_0^{SD} - p_{S0,I_S}, \]
\[ p_{L0,I_S} - \beta U_L^{ON} = W_1^{LD} - W_0^{LD} - p_{L0,I_S}, \]
\[ p_{L1,I_S} - \beta U_L^{ON} = W_1^{LD} - W_1^{LD} - p_{L1,I_S}. \]

2.4 Prices and Match Surpluses in the Investor-dealer Market

As we remarked earlier, in defining the value functions in (1)-(3), (6) and (7), and (11)-(13), we assume that there are non-negative match surpluses in any and all meetings in the investor-dealer market. A priori this need not be true as it is not inconceivable that an investor-buyer (investor-seller) may find it optimal to trade with one type of dealer-seller (dealer-buyer) but not others in equilibrium. By Lemma 1 below, the restriction is without loss of generality in any steady-state equilibrium with active trading.

**Lemma 1** In any steady-state equilibrium, the match surplus for meetings between an investor-seller and any dealer-buyer all equals to
\[ z_{I_S} = p - \beta U_L^{ON}, \]
while any such exchanges take place at the same price,
\[ p_{S0,I_S} = p_{L0,I_S} = p_{L1,I_S} = \frac{p + \beta U_L^{ON}}{2}. \]

The match surplus for meetings between an investor-buyer and any dealer-seller all equals to
\[ z_{I_B} = \beta (U_H^{ON} - U_B) - p, \]
while any such exchanges take place at the same price,

\[ p_{IB,S_1} = p_{IB,L_1} = p_{IB,L_2} = \frac{p + \beta (U^Q - U^B)}{2}. \]

(23)

In a narrow sense, the Lemma holds because of a competitive inter-dealer market in which dealers buy and sell the asset at the same price \( p \). In particular, if a dealer buying a unit from an investor finds it optimal to dispose it in the inter-dealer market right after, the dealer earns a surplus equal to \( p \) minus the price he pays to the investor, whereas if the dealer finds it optimal to keep the unit for selling to other investors later on, he earns just the same surplus as the unit would have cost him \( p \) in the inter-dealer market had he not bought it earlier from the investor. On the other hand, to the investor-seller, the gain from trade is the receipt of the selling price net of the continuation value of being a low-valuation owner, which is otherwise independent of the identity of the counterparty of the trade. The match surplus, equal to the sum of the surpluses from trade of the two sides, is then the same across all trades between an investor-seller and any dealer-buyer as given in (20). A similar logic explains (22). In a broader sense, as we shall show in Lemma 5 when we extend the analysis to a frictional inter-dealer market, that any trade between a dealer and an investor-seller should yield a non-negative surplus is that had the dealer found it not optimal to buy from an investor, the dealer would have to be buying the asset from another dealer, who had bought a unit from an investor earlier, for him to be an active agent in equilibrium. But then there cannot be any greater surplus for a unit to be acquired by one dealer first and then passed onto another dealer than for the unit to be bought by the latter dealer in the first instance.

2.5 Inter-dealer Market Trades

By (4) and (5), whether a small dealer entering the inter-dealer market wants to buy or sell depends on how the inter-dealer market price \( p \) compares with \( \beta (V^{SD}_1 - V^{SD}_0) \). Similarly, by (8)-(10), a large dealer entering the market decides to buy or sell by comparing \( p \) against \( \beta (V^{LD}_1 - V^{LD}_0) \) and \( \beta (V^{LD}_2 - V^{LD}_1) \). To proceed, we first establish that:

**Proposition 1** \( V^{LD}_1 - V^{LD}_0 \geq V^{SD}_1 - V^{SD}_0 \geq V^{LD}_2 - V^{LD}_1 \) in any active steady-state equilibrium in which (20) and (22) are non-negative. The first inequality is strict if (20) is strictly positive. The second inequality is strict if (22) is strictly positive.

Proposition 1 says that in an active steady-state equilibrium, an \( L_0 \) entering the inter-dealer market has the most to gain from acquiring a unit of the asset in the market, followed by an \( S_0 \), whereas an \( L_1 \) has to the least to gain. Intuitively, \( V^{LD}_1 - V^{LD}_0 \geq V^{SD}_1 - V^{SD}_0 \geq V^{LD}_2 - V^{LD}_1 \) because the opportunity cost for the large dealer in utilizing his first unit of inventory capacity should be lower than the opportunity cost for the small dealer in utilizing his only unit of inventory capacity – in acquiring a unit in the inter-dealer market, the large dealer, but not the small dealer, still has spare inventory capacity to buy one more unit from an investor in the next period to capture any possible surplus of trade. If the latter surplus is strictly positive, then a large dealer gains strictly more from the first unit of inventory than a small dealer does. When acquiring a unit in the inter-dealer market is at the expense of exhausting one’s
inventory capacity for both the large and small dealers, however, the small dealer should have
more to gain than the large dealer \((V_1^{SD} - V_0^{SD} \geq V_2^{LD} - V_1^{LD})\) since the large dealer holding
a one-unit inventory already has a unit for sale to investors in the upcoming period, whereas
the small dealer does not.

The ranking in Proposition 1 implies that \(L_d\) at least weakly prefer to buy and \(L_s\) at
least weakly prefer to sell in equilibrium. Who else will buy and sell depends on what price
clears the inter-dealer market, a price that must be bounded by

\[
p \in [\beta (V_2^{LD} - V_1^{LD}) , \beta (V_1^{LD} - V_0^{LD})],
\]

since at any \(p\) above the upper bound of the interval, there can only be sellers and at any
\(p\) below the lower bound, there can only be buyers in the market. Besides, for any \(p\) not
exactly equal to \(\beta\) times one of the three marginal benefits in Proposition 1, any and all
dealers who desire to trade either strictly prefer to buy or sell. In this case, the market clears
only if the parameters conspire to just equate the measures of buyers and sellers. Such a
parameter configuration, however, can only make up a zero-measure subset of the parameter
space. Equilibrium obtains in general only for \(p\) just equal to \(\beta\left(V_1^{LD} - V_0^{LD}\right)\), \(\beta\left(V_1^{SD} - V_0^{SD}\right)\),
or \(\beta\left(V_2^{LD} - V_1^{LD}\right)\), at which there is one type of dealer holding a given inventory indifferent
between selling and not selling or between buying and not buying. The market may then clear
at some particular mixing probability for the mixed strategy played by the marginal buyers or
sellers. Furthermore, we can show that:

**Lemma 2** For \(p\) equal to \(\beta\left(V_1^{LD} - V_0^{LD}\right)\) or \(\beta\left(V_1^{SD} - V_0^{SD}\right)\), both (20) and (22) and \(p\) itself
are strictly positive, whereas for \(p\) equal to \(\beta\left(V_2^{LD} - V_1^{LD}\right)\), (20) and \(p\) itself are equal to zero
while (22) is strictly positive. In all cases, the candidate equilibria are active equilibria in which
the gains from trade between investors and dealers are non-negative.

Let us first suppose that \(p = \beta\left(V_2^{LD} - V_1^{LD}\right)\), in which case an \(L_1\) entering the inter-dealer
market feels indifferent between paying \(p\) to buy one more unit and not buying. If he does
buy, he exhausts his entire inventory capacity and the purchase would be at the expense of
giving up the opportunity to buy from an investor in the next period. Meanwhile, given that
he already possesses a unit in inventory to begin with, he can sell to an investor in the next
period without buying a unit in the inter-dealer market. Thus, the dealer must be worse off
acquiring a unit at any positive \(p\). With \(p = 0\), a dealer is willing to buy a unit from an investor
also only at a zero price, which means that there must be but a zero surplus in a dealer-buyer
and investor-seller trade. If the latter gains from holding a unit of the asset. In the present setting, the
investor-seller does not have direct access to trading with an investor-buyer but can only trade with a dealer,
who may possibly not gain from acquiring the unit in which case the trade can only take place at a zero price.

In the other two cases, if \(p\) were equal to zero, then \(V_1^{SD} = V_0^{SD}\) must hold, meaning
that a small dealer earns the same expected trade surplus searching as either a dealer-buyer

\[9\]
or dealer-seller. At $p = 0$, however, as we argue above, any dealer must be earning just a zero trade surplus as a buyer and a positive trade surplus as a seller in the investor-dealer market. There must then be a strictly positive price in the inter-dealer market.

It is useful to classify equilibrium into three types, corresponding to $p$ equal to each candidate equilibrium price.

The “Selling” Equilibrium In the Selling Equilibrium, $p = \beta (V_1^{LD} - V_0^{LD})$. By Proposition 1 and Lemma 2,

$$p = \beta (V_1^{LD} - V_0^{LD}) > \beta (V_1^{SD} - V_0^{SD}) > \beta (V_2^{LD} - V_1^{LD}),$$

in which case no dealers strictly prefer to buy, whereas any dealers, large and small, with a filled inventory strictly prefer to sell. For this reason, we call this the Selling Equilibrium in which the optimal inventory level of a large dealer is zero or one unit, whereas that of a small dealer is zero unit. Even though $L_0$s are indifferent between buying and not buying, at least a fraction of them must buy in equilibrium since they are the only possible buyers. The market clears with a particular fraction of $L_0$s each buying a unit and possibly a certain fraction of $L_1$s each selling a unit when the net volume of purchase among the $L_0$s and $L_1$s is equal to the supply out of all $S_1$s and $L_2$s choosing to sell a unit each. Given that any equilibrium trading pattern should facilitate all dealers reaching their respective optimal inventories, the equilibrium with the least transactions, in which all $L_1$s refrain from selling so that all purchases made by $L_0$s are for meeting the supply from the inframarginal sellers, is arguably the most compelling one, among the continuum. Indeed, that equilibrium is the unique equilibrium if there is a vanishingly small, but positive, cost for each sale/purchase to be executed — an assumption we should maintain in the following — whereby, with no gains from trade, no $L_1$s will incur the trading cost to sell.\footnote{No $L_0$s will be willing to trade too if they are obliged to pay the transaction cost. The $L_2$s and $S_1$s, however, would be willing to finance the transactions with the $L_0$s since they strictly prefer to trade.}

The “Balanced” Equilibrium In the Balanced Equilibrium, $p = \beta (V_1^{SD} - V_0^{SD})$. By Proposition 1 and Lemma 2,

$$\beta (V_1^{LD} - V_0^{LD}) > p = \beta (V_1^{SD} - V_0^{SD}) > \beta (V_2^{LD} - V_1^{LD}),$$

from which it follows that $L_0$s strictly prefer to buy one unit while $L_2$s strictly prefer to sell one unit. We refer to this as the Balanced Equilibrium, in which the optimal inventory level of a large dealer is one unit, whereas that of a small dealer is zero or one unit. For the inter-dealer market to clear, if large dealers selling (buying) outnumber large dealers buying (selling) in the market, the net demand (supply) by small dealers must just suffice to meet the net supply (demand) from large dealers. With a small but positive cost of trade, a fraction of $S_0$s buy, while no $S_1$s sell, while no $S_0$s buy otherwise in the unique equilibrium.
The “Buying” Equilibrium  In the Buying Equilibrium, \( p = \beta (V_2^{LD} - V_1^{LD}) \). By Proposition 1 and Lemma 2,

\[
\beta (V_1^{LD} - V_0^{LD}) = \beta (V_1^{SD} - V_0^{SD}) > \beta (V_2^{LD} - V_1^{LD}) = p = 0,
\]

from which it follows that no dealers strictly prefer to sell. Meanwhile, any dealers with an empty inventory, large and small, strictly prefer to buy. For this reason, we call this the Buying Equilibrium in which the optimal inventory level of a large dealer is one or two units, whereas that of a small dealer is one unit. For the inter-dealer market to clear, a fraction of \( L_2 \) sell since they are the only possible sellers, while all \( L_1 \)s refrain from buying if trading is costly.

Discussions

Gains from inter-dealer trades  Gains from trade in our model arise (1) out of dealers not having continuous access to trading among themselves and (2) from dealers having different inventory capacities. If dealers do not have continuous access to the inter-dealer market, they adjust their inventories (or equivalently spare inventory capacities) every time the inter-dealer market opens to prepare themselves for future buying or selling opportunities with investors. If dealers possess different inventory capacities, their optimal inventories differ. In this environment, there can be a mutually beneficial trade between a large and a small dealer when one dealer’s inventory falls short of while the other dealer’s inventory exceeds their respective optimal inventories. There can also be mutually beneficial trades between two large dealers, as the gain to an \( L_0 \) from acquiring the first unit of the asset exceeds the loss suffered by an \( L_2 \) in forgoing the last unit. On the other hand, there can never be any gain from trade between two small dealers – an \( S_1 \) selling to an \( S_0 \) merely results in the two dealers switching states.

Identities of buyers and sellers  In Table 1, we summarize the optimal inventories of large and small dealers upon exiting and the identities of the buyers and sellers upon entry into the inter-dealer market, where the second line of each cell of the “Buyers” and “Sellers” columns indicate the identities of the marginal buyers and sellers in the three types of equilibrium.

Table 1 shows that large dealers with a one-unit inventory never buy or sell in the inter-dealer market. That is, a one-unit inventory is optimal for a large dealer in any equilibrium. The reason is that a dealer holding a one-unit inventory as well as possessing a unit of spare inventory capacity is able to take advantage of all future trading opportunities with investors given that the dealer meets at most one investor-buyer and one investor-seller per period. Table 1 also shows that at least a fraction of \( L_0 \)s buy, whereas at least a fraction of \( L_2 \)s sell in any equilibrium. What differs among the equilibria is the role played by small dealers. In the Selling Equilibrium, \( S_1 \)s sell while \( S_0 \)s stay out of the market. In the Buying Equilibrium, \( S_0 \)s buy while \( S_1 \)s stay out of the market. In the Balanced Equilibrium, small dealers may either sell or buy, depending on whether or not the buyers among large dealers outnumber the sellers.
Equilibrium | Price | Optimum Inventory | Buyers | Sellers  
---|---|---|---|---  
Selling | \( p = \beta (V_1^{LD} - V_0^{LD}) \) | \( 0 \) and \( 1 \) and \( 0 \) | \( L_0 \) | \( L_2, S_1 \)  
Balanced | \( p = \beta (V_1^{SD} - V_0^{SD}) \) | \( 1 \) | \( L_0 \) | \( S_1 \)  
Buying | \( p = \beta (V_2^{LD} - V_1^{LD}) \) | \( 1 \) and \( 2 \) | \( S_0, L_0 \) | \( L_2 \)  

Table 1: Prices, Optimal Inventories, Buyers and Sellers in Equilibrium

**Core-periphery trading network**  The trading pattern then resembles a core-periphery trading network, in which large dealers, each of which trades with all dealers, can be thought of as in the core of network, whereas small dealers can be thought of as in the periphery of the network, in that they only trade with large dealers. The prediction is consistent with the empirical findings documented in Li and Schürhoff (2014) and Hollifield, Neklyudov and Spatt (2016).11

**The persistence of the direction of trade**  In any type of equilibrium in our model, a large dealer sells to another large dealer at a point in time when the first dealer happens to possess a filled inventory while the other dealer happens to possess an empty inventory. At other times, the two dealers may switch roles when each happens to possess the opposite inventory. All this means that there is no persistent trading direction among large dealers. In a given type of equilibrium, however, small dealers either always sell to or buy from large dealers. Now, if only one type of equilibrium can hold for a given parameter configuration – a result we will establish in Proposition 2 to follow – the direction of trade between small and large dealers is persistent. The implication is also consistent with the findings in Li and Schürhoff (2014) that given that there is a directional (buy or sell) trade between two dealers in one month, the probability that the same directional trade remains in the next month is 62%.

**Difference in inventory**  Another empirical finding in Li and Schürhoff (2014) is that dealers in the core of the inter-dealer trading network hold more assets in inventory. In all three types of equilibrium in our model, the optimal inventory level for a large dealer (who is in the core of the network) is at least weakly higher than that of a small dealer (who is in the periphery), given that the opportunity cost in utilizing the first unit of inventory capacity is lower for the large dealer than that for the small dealer in utilizing his only unit of inventory capacity.

Our major results thus far – Proposition 1 and the implications thereof – seemingly rest on a number of questionable assumptions. First, if small dealers can only hold at most one

---

11Given that the inter-dealer market is assumed to be a Walrasian market, there is of course no prediction as to whom a given dealer is selling to or buying from. Where small dealers who trade in the inter-dealer market are either all sellers or all buyers, it is by no means far-fetched to say that a trading small dealer cannot be selling to or buying from another small dealer.
unit of the asset in inventory, it may seem trivial that they do not gain from trading among themselves. An obvious question to ask is how the core-periphery trading structure may hold if small dealers do gain by trading among one another in case they each possess more than a unit inventory capacity. Second, an important lesson in Li and Schürhoff (2014) and the follow-up study in Henderschott, Li, Livdan and Schürhoff (2016) is that the inter-dealer market is itself a decentralized market as opposed to a Walrasian market. Finally, if large dealers can hold up to two units in inventory and may possess up to two units of spare inventory capacity, perhaps a more natural assumption is that they can meet up to two investor-buyers and two investor-sellers in each period. We will discuss how Proposition 1 and its main implications survive all three generalizations in Section 4 below.

A given type of equilibrium places a set of restrictions on the measures of dealer-buyers, dealer-sellers and the asset held by these dealers, which in turn impact on the probabilities at which investors and dealers buy and sell in the investor-dealer market. Then, a candidate equilibrium can indeed be equilibrium only if these restrictions are met and where the probabilities of trades are bounded below one, in addition to the existence of some positive mixing probability for the marginal buyers’ or sellers’ mixed strategy which clears the inter-dealer market. We now proceed to study the underlying environment as defined by the asset supply $A$, the turnover rate of high-valuation owners $\delta$, the entry rate of high-valuation non-owners $e$, and the measures of large and small dealers in which the restrictions of each type of equilibrium are met.

2.6 Accounting Identities, Market Tightness, and Stock-Flow Equations

At the beginning of the first subperiod, if the market is populated by $n^{SD}$ small dealers and $n^{LD}$ large dealers,

\begin{align*}
n_0^{SD} + n_1^{SD} &= n^{SD}, \\
n_0^{LD} + n_1^{LD} + n_2^{LD} &= n^{LD}.
\end{align*}

The asset is in fixed supply equal to $A$, and hence,

\begin{equation}
n_H^{ON} + n_L^{ON} + n_1^{SD} + n_1^{LD} + 2n_2^{LD} = A,
\end{equation}

where $n_H^{ON}$ and $n_L^{ON}$ denote, respectively, the measures of high-valuation owners and low-valuation owners.

Let $n_B^I$ denote the measure of high-valuation non-owners $cum$ investor-buyers. Then,

\begin{equation}
\theta_{ID} = \frac{n_B^I}{n_S^D} = \frac{n_B^I}{n_1^{SD} + n_1^{LD} + n_2^{LD}}.
\end{equation}

Recall that the population of investor-sellers is comprised of the low-valuation owners. Then,

\begin{equation}
\theta_{DI} = \frac{n_B^D}{n_L^{ON}} = \frac{n_0^{SD} + n_0^{LD} + n_1^{LD}}{n_L^{ON}}.
\end{equation}
In the steady state, the respective inflows and outflows of high-valuation owners, low-valuation owners, and investor-buyers are equal. Hence,

\[
I_B \mu (\theta_{ID}) = \delta n_H^Q N ,
\]

\[
\delta n_H^Q N = \eta (\theta_{DI}) n_L^Q N ,
\]

\[
e = n_I^I \mu (\theta_{ID}) .
\]

Not all \( n_i^{SD} \) and \( n_i^{LD} \) can be positive in a given type of equilibrium. In the Selling Equilibrium for example, by the third and fourth columns of Table 1, all small dealers exit the inter-dealer market with an empty inventory, whereas large dealers may do so with either an empty or a one-unit inventory. The restrictions on the measures of small and large dealers with various levels of inventory in the three types of equilibrium are as depicted in Table 2.

<table>
<thead>
<tr>
<th>( n_i^{SD} )</th>
<th>( n_i^{LD} )</th>
<th>( \text{Selling} )</th>
<th>( \text{Balanced} )</th>
<th>( \text{Buying} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_0^{SD} )</td>
<td>( n_1^{SD} )</td>
<td>0, ( n_i^{SD} )</td>
<td>0, ( n_i^{SD} )</td>
<td>( n_i^{SD} )</td>
</tr>
<tr>
<td>( n_0^{LD} )</td>
<td>0, ( n_1^{LD} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n_1^{LD} )</td>
<td>0, ( n_2^{LD} )</td>
<td>( n_i^{LD} )</td>
<td>0, ( n_i^{LD} )</td>
<td></td>
</tr>
<tr>
<td>( n_2^{LD} )</td>
<td>0</td>
<td>0</td>
<td>0, ( n_i^{LD} )</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Measures of dealers entering the investor-dealer market in equilibrium

Given the measures of dealers, \( n_i^{SD}, i = 0, 1, \) and \( n_i^{LD}, i = 0, 1, 2, \) when the investor-dealer market opens in the first subperiod, the corresponding measures of dealers leaving the market and entering the inter-dealer market in the second subperiod, denoted as \( m_i^{SD}, i = 0, 1, \) and \( m_i^{LD}, i = 0, 1, 2, \) are given by the following.

\[
m_0^{SD} = (1 - \mu (\theta_{DI})) n_0^{SD} + \eta (\theta_{ID}) n_1^{SD},
\]

\[
m_1^{SD} = \mu (\theta_{DI}) n_0^{SD} + (1 - \eta (\theta_{ID}) ) n_1^{SD},
\]

\[
m_0^{LD} = (1 - \mu (\theta_{DI})) (n_0^{LD} + \eta (\theta_{ID}) n_1^{LD}),
\]

\[
m_1^{LD} = \mu (\theta_{DI}) n_0^{LD} + [\mu (\theta_{DI}) \eta (\theta_{ID}) + (1 - \mu (\theta_{DI})) (1 - \eta (\theta_{ID})) ] n_1^{LD} + \eta (\theta_{ID}) n_2^{LD},
\]

\[
m_2^{LD} = (1 - \eta (\theta_{ID})) (\mu (\theta_{DI}) n_1^{LD} + n_2^{LD}).
\]

For example, (32) says that \( S_0 \)'s entering the inter-dealer market are among the \( S_0 \)'s entering the investor-dealer market who fail to buy a unit in the market and the \( S_1 \)'s who succeed in selling the unit they each possess.

2.7 Market clearing in the Inter-dealer Market

**Selling Equilibrium** In the Selling Equilibrium, all dealers entering the inter-dealer market with a filled inventory strictly prefer to sell, whereas the only dealers who weakly prefer to buy are \( L_0 \)'s. For the inter-dealer market to clear, the measure of dealers who strictly prefer to sell must not exceed the measure of dealers who weakly prefer to buy; i.e.,

\[
m_1^{SD} + m_2^{LD} \leq m_0^{LD} .
\]
**Balanced Equilibrium** In the Balanced Equilibrium, $L_0$s entering the inter-dealer market strictly prefer to buy, whereas $L_2$s strictly prefer to sell. Then, if the latter outnumber the former; i.e.,

$$m_{2L} \geq m_{0L}, \quad (38)$$

a fraction of $S_0$s entering the inter-dealer market must buy in equilibrium; otherwise, a fraction of $S_1$s must sell in equilibrium. Hence, in case (38) holds, the inter-dealer market clears if

$$m_{2L} - m_{0L} \leq m_{0S}; \quad (39)$$

otherwise the inter-dealer market clears if

$$m_{0L} - m_{2L} \leq m_{1S}. \quad (40)$$

**Buying Equilibrium** In the Buying Equilibrium, all dealers entering the inter-dealer market with an empty inventory strictly prefer to buy, whereas the only dealers who weakly prefer to sell are $L_2$s. For the inter-dealer market to clear, the measure of dealers who strictly prefer to buy must not exceed the measure of dealers who weakly prefer to sell; i.e.,

$$m_{0S} + m_{2L} \leq m_{2L}. \quad (41)$$

2.8 Equilibrium

Given $\{n^{SD}, n^{LD}, A, e, \delta\}$, a steady-state equilibrium consists of the respective non-negative values of $n_0^{SD}, n_0^{LD}, n_1^{LD}, n_2^{LD}, n_H^{QN}, n_L^{QN}$ and $n_B$ that satisfy (24)-(31), the restrictions on $n_i^{SD}$ and $n_i^{LD}$ in Table 2 and the market-clearing conditions for the type of equilibrium under consideration in (37)-(41). Write $n^D = n^{SD} + n^{LD}$ as the total measure of dealers.

**Proposition 2** The Selling Equilibrium and the Buying Equilibrium may only hold for $e < n^{LD}$. The Balanced Equilibrium may only hold for $e < n^{LD} + n^{SD}$.

a. For $e < n^{LD}$, define

$$B_S \equiv e + \frac{n^D}{\mu^{-1}\left(\frac{e}{n^D}\right)},$$

$$B_M \equiv n^{LD} + \frac{n^D}{\mu^{-1}\left(\frac{e}{n^D}\right)},$$

$$B_L \equiv n^D + \frac{n^{LD}}{\mu^{-1}\left(\frac{e}{n^D}\right)},$$

where $B_S \leq B_M \leq B_L$.

(i) for $A - e/\delta \in (B_S, B_M]$, the Selling Equilibrium holds,

(ii) for $A - e/\delta \in [B_M, B_L]$, the Balanced Equilibrium holds,

(iii) for $A - e/\delta \geq B_L$, the Buying Equilibrium holds.
b. For \( e \in [n^{LD}, n^{LD} + \frac{n^{SD}}{2}] \), the Balanced Equilibrium exists if
\[
A - \frac{e}{\delta} > e + \frac{n^{D} + n^{LD} - e}{\mu^{-1} \left( \frac{e}{n^{D} + n^{LD} - e} \right)} \equiv B_M.
\]

c. In the Balanced Equilibrium,

(i) for \( A - e/\delta < S \), small dealers sell in equilibrium,
(ii) for \( A - e/\delta > S \), small dealers buy in equilibrium,
(iii) for \( A - e/\delta = S \), small dealers do not trade in the inter-dealer market, where

\[
S \equiv n^{LD} + \frac{n^{SD}}{2} + \frac{n^{LD} + n^{SD}}{\mu^{-1} \left( \frac{e}{n^{LD} + \frac{n^{SD}}{2}} \right)}.
\]

For \( e < n^{LD} \), \( B_M \leq S \leq B_L \), whereas for \( e \in \left[ n^{LD}, n^{LD} + \frac{n^{SD}}{2} \right) \), \( B_M \leq S \).

A steady-state equilibrium exists only if the necessary conditions on \( e \), the entry rate of investors, stated at the beginning of the Proposition, are met. These conditions arise because if a measure of \( e \) investors enter the market as high-valuation non-owners in each period, in the steady state, there have to be the same measure of \( e \) investors exiting high-valuation non-ownership after buying a unit and the same measure of \( e \) investors exiting low-valuation ownership and the market altogether after selling their units in the same period. All this requires that more than \( e \) dealer-sellers and more than \( e \) dealer-buyers are present in the market. In the Selling Equilibrium, only large dealers sell in the investor-dealer market, whereas in the Buying Equilibrium, only large dealers buy in the investor-dealer market. Then, for either type of equilibrium to exist, a necessary condition is that \( e < n^{LD} \). In the Balanced Equilibrium, however, a fraction of small dealers enter the investor-dealer market with an empty inventory and a fraction enter holding a one-unit inventory. Then, there will also be small dealers among both dealer-buyers and dealer-sellers and a steady-state equilibrium may exist even for \( e \geq n^{LD} \) but not for \( e \geq n^{LD} + \frac{n^{SD}}{2} \) since if more than one-half of all small dealers search as buyers (sellers), then there can only be fewer than one-half searching as sellers (buyers) in the investor-dealer market.

The conditions in Parts (a)-(c) of the Proposition can be interpreted as conditions on the inventory of the asset held by dealers,

\[
A^D \equiv A - n_H^{ON} - n_L^{ON} = A - \frac{e}{\delta} - \frac{n_B^{D}}{\mu^{-1} \left( \frac{e}{n_B^{H}} \right)},
\]

for each type of equilibrium to hold.\(^{12}\) In a given type of equilibrium, \( A^D \) is first of all bounded by the measures of dealers who may be holding inventory of the asset. For example, in the

\(^{12}\)In each period in the steady state, a measure of \( e \) investors enter high-valuation ownership, whereas each exits at the rate \( \delta \), from which \( n_H^{ON} = e/\delta \) follows. Meanwhile, with the measure of low-valuation owners selling in each period equal to \( e \), \( \eta(\theta_{DT}) n_L^{ON} = e \), from which \( n_L^{ON} = n_B^{H}/\mu^{-1} \left( \frac{e}{n_B^{H}} \right) \) obtains given \( \theta_{DT} = n_B^{D}/n_L^{ON} \).
Selling Equilibrium, small dealers do not hold any inventory while a fraction of large dealers may each hold a unit, in which case

\[ 0 \leq A^D \leq n^{LD}. \]

Moreover, in the steady state, on the one hand, dealers’ asset holding must be sufficiently plentiful for them to sell \( e \) units to the high-valuation non-owners. On the other hand, it must not exceed the level that would leave the dealers with insufficient spare inventory capacity to buy \( e \) units from the low-valuation owners. All together, \( A^D \) must be bounded by

\[ e < A^D < n^{LD} + n^D - e. \]  

(42)

It can then be shown that combining the two requirements yield the conditions in Parts (a)-(c).

Among the three types of equilibrium, the Selling Equilibrium, in which only a fraction of large dealers may hold just a one-unit inventory, involves dealers holding the least inventory, whereas the Buying Equilibrium, in which only a fraction of large dealers may still have one unit of spare inventory capacity, involves dealers holding the largest inventory. Part a(i) of the Proposition can be interpreted to say that when \( A^D \), held entirely by large dealers, just suffices to satisfy the demand from investor-buyers, the Selling Equilibrium begins to hold and it holds until \( A^D \) is up to the level at which all large dealers are holding a unit. At this point, according to Part a(ii), the Balanced Equilibrium begins to hold and it holds until \( A^D \) is up to the level at which all small dealers are holding a unit in inventory as well. Thereafter, by Part a(iii), the Buying Equilibrium holds, in which all dealers hold at least a one-unit inventory and a fraction of large dealers are holding a two-unit inventory. Part (b) of the Proposition, as in Part (a), says that a steady-state equilibrium exists once \( A^D \) is up to the level to satisfy investors’ asset demand.

Parts a(iii) and (b) of the Proposition say that a steady-state equilibrium holds even for arbitrarily large \( A - e/\delta \), which can be interpreted as how the upper bound on \( A^D \) in (42) will never be reached. In our model, as the asset supply increases and the sellers’ side of the investor-dealer market is becoming increasingly congested, it takes longer and longer for each low-valuation owner to sell, during which the measure of low-valuation owners and their asset holdings increase without bounds. With \( n^{ON} \) increasing in tandem with the asset supply, the inventory held by dealers never rises above the level that would leave them with insufficient spare inventory capacity to buy \( e \) units of the asset from investors.

In Part (c) of the Proposition, when \( A - e/\delta = S \) just holds in the Balanced Equilibrium, \( A^D \) is at the level at which dealers buy from and sell to investors at the same probability. Then, there would be just as many \( L_0 S \) and \( L_2 S \) entering the inter-dealer market, in which case small dealers do not trade in the market. For any smaller (larger) \( A^D \), dealers buy at a smaller (larger) probability while they sell at a higher (smaller) probability in the investor-dealer market to result in fewer (more) \( L_2 \) dealers entering the inter-dealer market to sell than \( L_0 \) dealers entering the market to buy. Small dealers sell (buy) in equilibrium to eliminate the excess demand (supply) among large dealers.

The proof of the Proposition in the Appendix shows that the condition in Parts (a)-(c) are also the conditions for how the inter-dealer market can clear for the given type of equilibrium in (37)-(41), which requires that there are fewer dealers who strictly prefer to trade in one
direction than there are dealers who weakly prefer to trade in the opposite direction. The reason is as follows. While all dealers at least weakly prefer to reverse the transactions with investors by trading in the inter-dealer market, those who strictly prefer to trade are those who need to regain their respective optimal inventories after trading with investors. These dealers must only be a subset of all dealers who sell to (buy from) investors given that there are two optimal inventory levels for a given type of dealers in each type of equilibrium.13 Now, in the steady state, there must be the same measure of dealers selling to and buying from investors in each period and thus it follows that dealers who weakly prefer to reverse the transactions with investors in one direction must be more numerous than those who strictly prefer to do so in the other direction. Thus, if the equilibrium conditions for the investor-dealer market of a given type of equilibrium in Parts (a)-(c) hold, the market-clearing condition for the inter-dealer market is guaranteed to be satisfied.

2.9 The Makeup of the Dealer Population and Equilibrium Types

Proposition 2 focuses on the role the asset supply plays in determining equilibrium outcomes. To complete the analysis, we next turn our attention to the role the makeup of the dealer population plays.

To begin, if there were no large dealers, any inter-dealer trades would only be between an $S_1$ selling to an $S_0$. The equilibrium terms of trade must then be such that the two parties are indifferent between trading and not trading; i.e. $p = \beta (V_{1}^{SD} - V_{0}^{SD})$ as in the Balanced Equilibrium. For any positive $n^{LD}$ not up to $e$, by Proposition 2, there can still only be a Balanced Equilibrium in which small dealers remain indifferent between trading and not trading. But as $n^{LD}$ rises up to and above $e$, the Selling and the Buying Equilibrium may begin to hold in which small dealers strictly prefer to trade.

The conditions in Proposition 2 may be manipulated to trace out the evolution of equilibrium type as $n^{LD}$ increases from the smallest admissible value to $n^{D}$.

**Corollary 1** Holding fixed $n^{D}$, the Balanced Equilibrium begins to hold for

$$n^{LD} > \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - e} \right) \left( A - \frac{e}{\delta} - e \right) - n^{D} + e.$$

As $n^{LD}$ increases from the lower bound in the above towards $n^{D}$, the Balanced Equilibrium holds throughout only if $A - e/\delta = \overline{B}$ holds exactly, where

$$\overline{B} = n^{D} + \frac{n^{D}}{\mu^{-1} \left( \frac{e}{n^{D}} \right)}.$$

In general, the Balanced Equilibrium changes into the Buying Equilibrium for $A - \frac{e}{\delta} > \overline{B}$ at

$$n^{LD} = \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - n^{D}} \right) \left( A - \frac{e}{\delta} - n^{D} \right),$$

---

13 A dealer who enters the investor-dealer market with an $i$-unit inventory and acquires one more unit from an investor does not strictly prefer to sell in the inter-dealer market afterwards if the dealer is indifferent between holding an $i$- and an $(i+1)$-unit inventory; contrariwise, a dealer who enters the investor-dealer market with an $i$-unit inventory and sells one unit to an investor does not strictly prefer to buy in the inter-dealer market if the dealer is indifferent between holding an $i$- and an $(i-1)$-unit inventory.

21
but into the Selling Equilibrium for $A - \frac{e}{\delta} < B$ at

$$n^{LD} = A - \frac{e}{\delta} - \frac{n^D}{\mu^{-1}(\frac{e}{\delta})}.$$ 

When large dealers become relatively more numerous then, the Balanced Equilibrium in general must give way to either the Buying or the Selling Equilibrium. In particular, if the asset supply is relatively abundant as with $A - e/\delta > B$, for the inter-dealer market to clear in the Balanced Equilibrium, $S_0$s buy to eliminate the excess supply among large dealers. But then when $n^{LD}$ rises up to and $n^{SD}$ falls down to some given levels, the remaining $S_0$s would no longer be sufficiently numerous to fulfill such a role. In this case, equilibrium can only obtain when $L_2$s no longer strictly prefer to sell as when the inter-dealer market price falls from $p = \beta (V_{1}^{SD} - V_{0}^{SD})$ to $p = \beta (V_{2}^{LD} - V_{1}^{LD})$, at which point the Buying Equilibrium takes hold. Conversely, how the Balanced Equilibrium changes into the Selling Equilibrium when the asset supply is relatively meager as with $A - e/\delta < B$ is due to how $L_0$s must no longer strictly prefer to buy when the remaining $S_1$s would no longer suffice to fill the gap between the supply and demand from large dealers as they make up a large enough fraction of the dealer population.

3 Peripheral Dealers Provide Immediacy for Core Dealers

In other network-theoretic models of inter-dealer trades with a core-periphery structure, the core dealers are either identified with dealers that sell to and thus provide inventory for peripheral dealers,14 dealers that generally provide immediacy for peripheral dealers by virtue of being more connected to other dealers,15 or dealers that simply tend to trade more frequently.16 Our model has more specific and arguably more subtle implications on the direction of trade between core and peripheral dealers.

In particular, a direct corollary of Proposition 2 is that:

**Corollary 2** Large dealers (holding a two-unit inventory) sell to small dealers (with an empty inventory) for $A > S + e/\delta$, whereas for $A < S + e/\delta$, large dealers (with an empty inventory) buy from small dealers instead.

In our model then, if the large dealers, who are in the core, are to be interpreted as providing inventory (liquidity) for small dealers, they do so when the asset supply is relatively abundant (meager), just when small dealers should need inventory (liquidity) the least.

Dealers need inventory more than spare inventory capacity in a market with a small asset supply—the scarcity of the asset should give rise to relatively more selling opportunities than buying opportunities for dealers in the investor-dealer market.17 The competitive inter-dealer

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14In Farboodi (2014), large banks in the core initiate risky investment projects and acquire funding from small banks in the periphery. In Zhong (2014), the core dealer acquires a risky asset from an investor and sells to other dealers to which the dealer is connected for risk-sharing.

15Wang (2017) being a notable example.

16For example, Neklyudov (2015) and Hugonniery et al. (2016).

17That is, it should be easier for dealers to meet investor-buyers than investor-sellers. This is established in Proposition 4a below.
market should then serve to allocate the inventory to the dealers who value them more \( (L_0s) \) over those who value them less \( (S_0s) \), by means of the sales from \( S_1s \) to \( L_0s \). On the other hand, dealers need spare inventory capacity more than inventory in a market with a large asset supply — the abundance of the asset should give rise to relatively more buying opportunities than selling opportunities for dealers in the investor-dealer market. The competitive interdealer market should then serve to allocate the spare capacity to dealers who value them more \( (L_2s) \) over those who value them less \( (S_1s) \), by means of the purchases by \( S_0s \) from \( L_2s \). All together, a more natural interpretation in our model is that it is the small peripheral dealers that provide inventory (liquidity) for the large core dealers when the latter need inventory (liquidity) the most. In general, the small peripheral dealers trade to provide immediacy for the large core dealers.

This implication is counterintuitive but there indeed exists preliminary empirical evidence supporting the implication. Hollifield et al. (2015) report in their Table 6 that while the percentage bid-ask spreads peripheral dealers earn when they are the first links of the intermediation chains, buying from an investor for selling to other dealers, is smaller than the percentage spreads they earn when they are the last links, buying from another dealer for selling to an investor, there are no statistically significant differences between the two spreads for core dealers.

In our model, whether a peripheral dealer is buying from or selling to other dealers and the magnitudes of the markups are both equilibrium outcomes as depending on the abundance of asset supply.

Specifically, when the latter part of the Balanced Equilibrium and the Buying Equilibrium hold as arising from an abundant asset supply, the small peripheral dealers are in the last link of the intermediation chains, providing liquidity for large dealers by buying from the latter, while earning the markup,

\[
\rho_{DS} \equiv \frac{p_{IB}}{p} - 1,
\]

in return. When the asset supply is abundant, however, the small peripheral dealers would be selling to investors in a slow investor market (small \( \theta_{ID} \)) — a market in which it takes a long time on average for a dealer to sell the unit in his inventory. The intermediation services provided by small dealers for large dealers by selling to investors on their behalf should then be particularly valuable, commanding a high return to commensurate with the liquidity services rendered in the slow market.

On the other hand, when the Selling Equilibrium and the earlier part of the Balanced Equilibrium hold as arising from a meager asset supply, small dealers are in the first link of the intermediation chains, providing inventory for large dealers by selling to the latter, earning the markup,

\[
\rho_{DB} \equiv \frac{p - p_{IS}}{p_{IS}},
\]

in return. When the asset supply is meager, the small peripheral dealers would be buying from investors in a tight market (large \( \theta_{DI} \)), a market populated by a large number of dealer-buyers versus a small number of investor-sellers. In the tight market, where the competition

\(^{18}\)We formally show that both \( \theta_{ID} \) and \( \theta_{DI} \) are (weakly) decreasing and continuous in \( A \) in Proposition 4a below.

23
among dealers is relatively intense, there can only be a small return earned by a given dealer from intermediating the sale by an investor as the investor should have plenty of meeting opportunities with the dealer’s competitors should a deal not be agreed upon.

Averaging over markets with various levels of asset supply, small dealers in our model indeed should earn a higher markup when they buy from other dealers for selling to investors than when they buy from investors for selling to other dealers, just as what Hollifield et al. (2015) find in their empirical analysis. To confirm this conjecture, we calculate and then plot $\rho_{DB}$ against $A$ when the Selling Equilibrium and the earlier part of the Balanced Equilibrium holds and $\rho_{DS}$ against $A$ when the latter part of the Balanced Equilibrium holds in FIG 1.\textsuperscript{19}

When the Buying Equilibrium holds, $\rho_{DS} = \infty$ with $p = 0$, and so is left out of the plot. With $\rho_{DS} = \infty$, a literal interpretation of the model suggests that if the Buying Equilibrium holds for at least one market in the sample, dealers should be observed to earn an infinite $\rho_{DS}$ on average, which is obviously not true. The implication is an artifact of assuming a zero flow payoff to low-valuation investors and dealers from holding a unit of the asset but not a generic feature of the model. The model’s $\rho_{DS}$ in the Buying Equilibrium can be made to become finite by assuming that low-valuation investors and dealers place a positive value, instead of zero, on holding a unit of the asset, in which case $p$ would stay positive in any equilibrium but still contrives to give rise to a relative large $\rho_{DS}$ in the Buying Equilibrium.

The tendencies that $\rho_{DB}$ should be low for small $A$ and $\rho_{DS}$ should be high for large $A$ arise out of the influences of market tightness on agents’ bargaining positions and should be common to most models of frictional OTC markets. Models that are built to be consistent with core dealers providing immediacy for small dealers would then likely to predict just the opposite of the findings in Hollifield et al. (2015).

Last but not least, we should mention that our model can also be consistent with the finding in Hollifield et al. (2015) that the central dealers do not tend to earn higher or lower percentage markups between buying from and selling to customers in a dealer chain. The large core dealers in our model are dealer-buyers as well as dealer-sellers in all three types of equilibrium. For each $A$ then, there are large dealers earning $\rho_{DS}$ from buying from other dealers for selling to investors and $\rho_{DB}$ from buying from investors for selling to other dealers. In FIG 2, we plot $\rho_{DS}$ and $\rho_{DB}$ for all levels of $A$. Although large dealers earn a higher $\rho_{DS}$ than $\rho_{DB}$ in the Balanced and in the Buying Equilibria, they earn a higher $\rho_{DB}$ instead in the Selling Equilibrium. Averaging over the $A$ for assets included in a given sample, there can be no statistically significant differences between the observed $\rho_{DS}$ and $\rho_{DB}$ for the large dealers.

\textsuperscript{19}The numerical analyses assume $\eta(\theta) = 1 - e^{-\theta}$, $c = 0.8$, $\delta = 0.1$, $\beta = 0.95$, $n^{LO} = 0.5$, $n^{SD} = 1$ and $A$ varying from 9.86 – the lower bound for the Selling Equilibrium to hold – and up. The equations for $p$, $\theta_{DI}$, and $\theta_{ID}$ for the Selling Equilibrium are given by (78), (87), and (89), respectively. The equations for $p$, $\theta_{ID}$, and $\theta_{DI}$ in the Balanced Equilibrium are given by (80), (100), and (101), respectively. The equations for $p_{IS}$ and $p_{IB}$ for the Selling and Balanced Equilibria are stated in Lemma A1 in the Appendix. The discontinuities in $\rho_{DS}$ and $\rho_{DB}$ are due to $p$ changing its anchor from one to another indifference conditions as one equilibrium type changes to another. See Proposition 4c below.
Figure 1: Small dealers’ percentage bid-ask spreads

Figure 2: Large dealers’ percentage bid-ask spreads
4 Robustness of the Equilibrium Features

The model we studied in this paper is clearly a very special model, with numerous important simplifying assumptions. In this section, we explain how our major results should survive three generalizations that are most warranted. Notwithstanding the discussions below falling short of full-fledged formal analyses in various occasions for the interest of brevity, the claims not derived from formal proofs in the following all appear to be intuitive extensions of the corresponding results in the main model. All lemmas and Proposition 3 below are formal results though the proofs of which are in the Appendix.

4.1 Larger Inventory Capacity for Small Dealers

Perhaps it seems trivial that, in our model, small dealers, in having just one unit of inventory capacity, never gain from trading among themselves. The question then is if and how the core-periphery trading relationship in our setup and its implications on trading directions in Corollary 2 survive the generalization where small dealers each possess more than a unit of inventory capacity and thereby may gain by trading with one another.

Consider, in particular, that the small dealers each possess a two-unit inventory capacity, while the large dealers each possess a three-unit inventory capacity. These larger capacities are relevant only if a dealer may buy and sell up to two units of the asset in a period. The simplest extension is to assume that there are two types of investors – small and large, where the former, comprising a fraction $\alpha$ of the investor population, may each hold either zero or one unit, whereas the latter, comprising the rest of the investor population, may each hold either zero or two units, and that dealers meet investors randomly independent of dealers’ types. In this environment, a dealer-seller (-buyer) holding a one-unit inventory (spare capacity) can only sell to (buy from) small investors who buy(sell) one unit of the asset at a time, whereas a dealer-seller (-buyer) holding at least a two-unit inventory (spare capacity) can sell to (buy from) large investors, who buy (sell) two units at a time, as well as small investors.

Now, a ranking of the marginal benefits of inventory similar to that in Proposition 1 should remain – an additional unit of the asset should be valued higher by a large dealer than by a small dealer at the same initial level of inventory for the two dealers since the former would retain a greater spare capacity for future buying needs than the latter in using up a unit of capacity for acquiring the unit. On the other hand, the large dealer should value an additional unit of the asset less than the small dealer when they start with the same spare capacity, as the former has a larger initial inventory than the latter beforehand. The ranking of the marginal benefits of inventory in Proposition 1 may then be generalized to

$$V_{1}^{LD} - V_{0}^{LD} \geq V_{1}^{SD} - V_{0}^{SD} \geq V_{2}^{LD} - V_{1}^{LD} \geq V_{2}^{SD} - V_{1}^{SD} \geq V_{3}^{LD} - V_{2}^{LD},$$

where the inter-dealer market clears in general only if $p$ is equal to $\beta$ times one of the above marginal values. We next proceed to inquire how the core-periphery trading structure survives, as well as how the trading directions between small and large dealers remain persistent, with $p$ equal to each of the candidate equilibrium value.

Case 1a $p = \beta(V_{1}^{LD} - V_{0}^{LD})$ The buyers in the inter-dealer market are comprised of a fraction of $L_0$s and the sellers are $S_1$s, $S_2$s, $L_2$s, and $L_3$s.
Case 1b  \( p = \beta (V_{1}^{SD} - V_{0}^{SD}) \) with a fraction of \( S_1 \) s selling in the inter-dealer market

The buyers in the inter-dealer market are all of \( L_0 \) s and the sellers are a fraction of \( S_1 \) s, and all of \( S_2 \) s, \( L_2 \) s and \( L_3 \) s.

In both (1a) and (1b), given that all small dealers who trade in the inter-dealer market (\( S_1 \) s and \( S_2 \) s) sell and they sell to \( L_0 \) s, small dealers trade to provide inventory for large dealers and the trading direction between small and large dealers is persistent. Also, since small dealers only sell in the inter-dealer market to large dealers but never trade among themselves, the trading relationship retains the core-periphery structure.

Case 2  \( p = \beta (V_{1}^{SD} - V_{0}^{SD}) \) with a fraction of \( S_0 \) s buying in the inter-dealer market

The buyers in the inter-dealer market are all of \( L_0 \) s and a fraction of \( S_0 \) s, and the sellers are \( L_2 \) s, \( S_2 \) s and \( L_3 \) s. When this type of equilibrium first starts to hold, the fraction of \( S_0 \) s who buy is arbitrarily close to zero. When this equilibrium turns into the equilibrium at \( p = \beta (V_{2}^{LD} - V_{1}^{LD}) \) so that all \( S_0 \) s are buying, as we shall demonstrate below in the next case, there would be as many \( S_0 \) s as \( S_2 \) s. In between, we conjecture that there remains fewer \( S_0 \) buyers than \( S_2 \) sellers. Then, on balance, small dealers are providing inventory for large dealers and the trading direction is largely persistent.

In the inter-dealer market, a buyer (\( L_0 \) or \( S_0 \)) can buy from an \( S_2 \), \( L_2 \) or \( L_3 \), and thus has three links; a seller (\( S_2 \), \( L_2 \) or \( L_3 \)) can sell to an \( L_0 \) or some of the \( S_0 \) s, and thus has two links. As we argue above, \( S_2 \) sellers, each of whom has two links, should be more numerous than \( S_0 \) buyers, each of whom has three links. Meanwhile, that the inter-dealer market clears at a relatively high \( p \) must arise out of a relatively small asset supply, in which case it should be easier for dealers to sell than to buy from investors to result in more \( L_0 \) buyers, each of whom has three links, than \( L_2 \) and \( L_3 \) sellers together, each of whom has two links, entering the inter-dealer market. Moreover, it can be shown that:

Lemma 3 In this equilibrium, the fraction of small dealers who are not trading in the inter-dealer market in a period is greater than the corresponding fraction for large dealers; i.e., \( \frac{(1-\phi)m_{SD}^{D}+m_{SD}^{PD}}{m_{SD}^{D}} > \frac{m_{LD}^{D}}{m_{LD}^{D}} \), where \( \phi \in [0,1] \) denotes the fraction of \( S_0 \) s who are trading in the market, from which it follows that a given large dealer is more likely to be trading in a period than a given small dealer does.

All together, a given large dealer should be observed to trade more often in the inter-dealer market and with a greater variety of dealers on average than a given small dealer does, whereby the trading relationship retains a core-periphery favor.

Case 3  \( p = \beta (V_{1}^{LD} - V_{2}^{LD}) \) The buyers in the inter-dealer market are all of \( L_0 \) s, \( S_0 \) s, and possibly a fraction of \( L_1 \) s. The sellers are \( S_2 \) s, \( L_3 \) s, and possibly a fraction of \( L_2 \) s. Given that all small dealers leave the inter-dealer market with one unit of inventory whereas large dealers do so with either one or two units of inventory, when the investor-dealer market opens, all dealers are dealer-sellers as well as dealer-buyers. Meanwhile, in each period in the steady state, there would be \( \alpha e \) measure of small investors and \( (1-\alpha) e \) measure of large investors buying, as well as selling. All this can be shown to imply that:

27
Lemma 4  (a) $\eta(\theta_{ID}) = \mu(\theta_{DI})$, (b) $m_0^{SD} = m_2^{SD}$, and (c) $m_0^{LD} = m_3^{LD}$.

Part (a) of the Lemma says that dealers meet investor-buyers and investor-sellers at the same probability in this equilibrium. What follows next is that there are equal measures of inframarginal buyers $(m_0^{SD} + m_0^{LD})$ and sellers $(m_2^{SD} + m_3^{LD})$, whereby the indifferent traders $L_1$s and $L_2$s do not trade in the inter-dealer market. This equilibrium then corresponds to the mid-point of the Balanced Equilibrium in the basic model at which $\eta(\theta_{ID}) = \mu(\theta_{DI})$ holds and that the indifferent traders $S_0$s and $S_1$s do not trade. Here, with small dealers selling and buying in the inter-dealer market equally numerous, they provide neither inventory nor liquidity for large dealers when the asset supply is at an intermediate level to indeed result in $p = \beta (V_2^{LD} - V_1^{LD})$ in equilibrium. There is not any apparent core-periphery structure either, as any dealer-buyer ($S_0$ or $L_0$) and any dealer-seller ($S_2$ or $L_3$) both have two links. This equilibrium should hold only with just one level of $A$ to satisfy $\eta(\theta_{ID}) = \mu(\theta_{DI})$, as in the basic model.

Case 4  $p = \beta (V_2^{SD} - V_1^{SD})$ with a fraction of $S_2$s selling in the inter-dealer market

This is the mirror opposite of case 2. Small dealers buy from more than they sell to large dealers, providing liquidity for large dealers on balance.

Case 5  $p = \beta (V_2^{SD} - V_1^{SD})$ with a fraction of $S_1$s buying in the inter-dealer market or $p = \beta (V_3^{LD} - V_2^{LD})$  The is the mirror opposite of case 1. Small dealers buy from large dealers only, providing liquidity for large dealers.

The above suggests that the main results of our paper should also generalize to where there are more than two inventory capacities, as similar mechanisms should be operative to give rise to smaller-capacity dealers providing inventory (liquidity) for larger-capacity dealers when inventory (liquidity) is in greater demand. The inter-dealer trading network should retain a core-periphery favor as well since higher-capacity dealers should have more profitable trading opportunities with dealers of the same capacity than lower-capacity dealers have.

4.2 Frictional inter-dealer market

In reality, the inter-dealer market is better described as a decentralized market as suggested by the findings in Li and Schürhoff (2014) and Henderschott, Li, Livdan and Schürhoff (2016), where it takes time and effort for a dealer to find a counterparty to trade with, in which case dealers, by all means, have incentives to manage inventory for future trading needs. By assuming dealers only have periodic, instead of continuous, access to the competitive inter-dealer market, the dealers in our model likewise have incentives to manage inventory. Where the incentives are similar, many features of the equilibrium in the present model, such as the core-periphery inter-dealer trading structure, should survive in an arguably richer model of a frictional inter-dealer market.

In the following, we report the results of our analysis of a model of a frictional inter-dealer market, which is otherwise identical to the main model of the paper, except that the model is set in continuous time as it is a more convenient setting to analyze a model in which both
the investor-dealer and the inter-dealer markets are decentralized. In the revised model, we continue to assume that the search and matching in the investor-dealer market takes place in two market segments, with respective market tightness \( \theta_{ID} \) and \( \theta_{DI} \). The inter-dealer market is frictional, however, in which a given dealer meets another randomly selected dealer at a fixed rate \( \alpha \) per unit of time, and where the terms of trade between two dealers are determined by Nash Bargaining, as are prices in the investor-dealer market. The value functions and equilibrium conditions are presented in Appendix 7.1. The pricing equations are standard and are omitted for brevity.

4.2.1 Core-periphery Trading Structure

As a stepping stone to show that a core-periphery structure remains, we first verify that:

**Lemma 5** Any investor-dealer match yields a non-negative surplus in any equilibrium in which both small and large dealers are active.

This Lemma generalizes Lemma 1 for a competitive inter-dealer market. The proof of the Lemma proceeds with the idea that first, if a given dealer-seller \( DS \) chooses not to sell to investor-buyers (IB), he must then sell to other dealers, say dealer \( d \), for otherwise the unit will never be passed on to an IB, in which case it would never be optimal for dealer \( DS \) to acquire the unit in the first place. Now, if it is optimal for dealer \( d \) to sell to an IB, it must be optimal for dealer \( DS \) to sell to the IB as well – there cannot be any greater surplus of trade for the unit to pass to another dealer before the unit is sold to an IB. A similar argument explains how a given dealer possessing spare capacity must find it optimal to buy from an investor-seller should the opportunity arise if the dealer should stay active in equilibrium at all. Given the Lemma, we then proceed to show that the counterpart to Proposition 1 for the competitive inter-dealer market holds.

**Proposition 3** \( V_{1L}^{LD} - V_{0L}^{LD} \geq V_{1SD}^{SD} - V_{0SD}^{SD} \geq V_{2SD}^{LD} - V_{1LD}^{LD} \) in any active equilibrium. The two equalities are strict unless the surplus for the IB-L1 match,

\[
z_{IB,L1} = U_B^{ON} - U_B - (V_{1L}^{LD} - V_{0L}^{LD})
\]

is equal to zero.

By Proposition 3 and if the inequalities are strict, among the respective surpluses for the possible inter-dealer trades, only

\[
z_{L0,S1} = V_{1L}^{LD} - V_{0L}^{LD} + V_{0SD}^{SD} - V_{1SD}^{SD} > 0,
\]

\[
z_{S0,L2} = V_{1SD}^{SD} - V_{0SD}^{SD} + V_{1L}^{LD} - V_{2L}^{LD} > 0,
\]

\[
z_{L0,L2} = V_{1L}^{LD} - V_{0L}^{LD} + V_{1L}^{LD} - V_{2L}^{LD} > 0,
\]

whereas all other inter-dealer trades yield negative surpluses. Hence, exchanges between an \( L_0 \) and an \( S_1 \), between an \( S_0 \) and an \( L_2 \), and between an \( L_0 \) and an \( L_2 \) exhaust all profitable exchanges among dealers, as in the competitive inter-dealer market model, from which

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20 All notations for the revised model have the same meanings as for the main model.
4.2.2 Trading Directions

With a competitive inter-dealer market, we have shown that, in Corollary 2, small dealers only sell to (buy from), but not buy from (sell to), large dealers when the asset supply is relatively meagre (abundant). Apparently, such implications no longer hold with a frictional inter-dealer market as both types of trade, each yielding a positive surplus, would take place in equilibrium for all levels of asset supply. Even so, a weaker version of the Corollary can hold if the market as both types of trade, each yielding a positive surplus, would take place in equilibrium.

To begin, since, by Proposition 3, among all dealers, $S_1$s only sell to $L_0$s whereas $S_0$s only buy from $L_2$s, in the steady state in which $n_0^{SD} = n_1^{SD} = 0$,

$$n_1^{SD}(\eta(\theta_{ID}) + \alpha \frac{n_0^{LD}}{n_D}) = n_0^{SD}(\mu(\theta_{ID}) + \alpha \frac{n_2^{LD}}{n_D}).$$

(43)

Also, where $L_1$s do not trade in the inter-dealer market and that $L_0$s buy from $S_1$s and $L_2$s, whereas $L_2$s sell to $S_0$s and $L_0$s, the equations for $n_0^{LD} = 0$ and $n_2^{LD} = 0$ specialize to, respectively,

$$n_1^{LD}(\eta(\theta_{ID}) = n_0^{LD}(\mu(\theta_{DI}) + \alpha \frac{n_1^{SD} + n_2^{LD}}{n_D}),$$

(44)

$$n_2^{LD}(\eta(\theta_{ID}) = n_2^{LD}(\eta(\theta_{ID}) + \alpha \frac{n_0^{LD} + n_2^{LD}}{n_D}).$$

(45)

Given $\{n^{SD}, n^{LD}, A, e, \delta\}$, a steady-state equilibrium consists of the respective non-negative values of $n_0^{SD}, n_1^{SD}, n_0^{LD}, n_1^{LD}, n_2^{LD}, n_H^{ON}, n_L^{ON}$ and $n_B^{ID}$ that satisfy the accounting identities in (24)-(26) and the steady-state conditions for $n_H^{ON}, n_L^{ON}$ and $n_B^{ID}$ in (29)-(31), which are common for both the competitive and frictional inter-dealer market models, and (43)-(45) above. Appendix 7.1 shows that the said conditions can be consolidated into two equations in $\{\theta_{ID}, \theta_{DI}\}$. These two equations, however, turn out to be highly nonlinear in $\{\theta_{ID}, \theta_{DI}\}$, whereby it does not seem possible to derive analytically conditions on model fundamentals for the existence and uniqueness of equilibrium. Assuming a Walrasian inter-dealer market simplifies considerably and enables us to derive a rich set of analytical results.

In our numerical analyses, with $\eta(\theta) = \theta^{0.5}$, $\alpha = 1$, $e = 0.1$, $d = 0.05$, and $\{n^{SD}, n^{LD}\} = \{0.6, 0.1\}$, we first find that a unique steady-state equilibrium exists for $A \in [2.77, 3.61]$. Second and more importantly, we find that small dealers’ sales to large dealers or the volume of $L_0$-$S_1$ trades, given by

$$SD_s = n_1^{SD} \alpha \frac{n_0^{LD}}{n_D},$$
Figure 3: Small-large dealers trade volume

is decreasing in $A$, whereas small dealers’ purchases from large dealers or the volume of $S_0-L_2$ trades, given by

$$SD_b = n_0^{SD} \alpha \frac{n_2^{LD}}{n_D}$$

is increasing in $A$ as shown in FIG 3, and that $SD_a$ dwarfs (is dwarfed by) $SD_b$ for small (large) $A$. The point is shown more forcefully in FIG 4, which shows how the ratio $SD_a/SD_b$ declines from an arbitrarily large value to practically zero as $A$ rises from the lower to the upper bounds of the relevant interval, meaning that the great majority of trades between small and large dealers are sales from small to large dealers for relatively small $A$ and vice versa. \(^{22}\) These results are from assuming a large $n^{SD} (= 0.6)$ relative to $n^{LD} (= 0.1)$. As checks for robustness, we find the same qualitative results hold with \(\{n^{SD}, n^{LD}\} = \{0.1, 0.6\}\) and \(\{0.35, 0.35\}\).

To understand these results, start with the value of $A$ at which $SD_a = SD_b$; i.e.,

$$n_1^{SD} n_0^{LD} = n_0^{SD} n_2^{LD}, \quad (46)$$

and let $A$ go up by a small amount. Then, for $SD_a < SD_b$, as shown in FIG 3, to follow,

$$(n_1^{SD} + dn_1^{SD}) (n_0^{LD} + dn_0^{LD}) < (n_0^{SD} + dn_0^{SD}) (n_2^{LD} + dn_2^{LD}).$$

Expanding and given (46), the condition becomes,

$$n_1^{SD} dn_0^{LD} + (n_0^{LD} + dn_0^{LD}) dn_1^{SD} < n_0^{SD} dn_2^{LD} + (n_2^{LD} + dn_2^{LD}) dn_0^{SD}. \quad (47)$$

\(^{22}\)The bounds $A \in [2.77, 3.61]$ are not exact bounds, but are bounds that we found by rounding off to two decimal places. At the true lower bound, a value a bit below 2.77, $SD_a$ should be exactly equal to 0 with $n_2^{LD} = 0$. At the true upper bound, a value a bit above 3.61, $SD_a$ should be exactly equal to 0 with $n_0^{LD} = 0$. 31
In our numerical analyses, we find that, not surprisingly, $n_{SD}^0$ and $n_{LD}^0$ are everywhere decreasing whereas $n_{SD}^1$ and $n_{LD}^2$ are everywhere increasing in $A$ – when the asset is more abundant, fewer dealers are left with an empty inventory and more dealers are operating with a full inventory in equilibrium. Less obvious is that $n_{LD}^1$ is found to be decreasing in $A$ for $A$ not less than the level at which $SD_s = SD_b$ for all three tuples of $\{n_{SD}, n_{LD}\}$ that we tried out. Given that, by (24), (25), (27), and (28),

$$n_{LD}^1 = n_B^D + n_S^D - n_D^D,$$

and that an increase in dealers’ overall inventory holding, resulting from more units of the asset on the market, should lead to there being fewer dealer-buyers who still have capacity to buy from investors and more dealer-sellers holding at least a one-unit inventory to sell to investors, $n_{LD}^1$ declines if $n_B^D$ falls by more than the increase in $n_S^D$. But this is indeed what should tend to happen because of the frictions in the inter-dealer market. Transactions between dealers holding a full inventory and dealers with an empty inventory would serve to average out inventory holdings among dealers, lessening the decline in $n_B^D$ while leaving the increase in $n_S^D$ by just about the same amount as if there were no such transactions. In case such transactions are prevented from taking place to the fullest extent possible due to the trading frictions in the inter-dealer market, $n_B^D$ can fall by a considerably greater amount to give rise to an overall smaller $n_B^D + n_S^D$.

Now, given that (1) $dn_{LD}^2 = -dn_{BD}^L - dn_{0D}^L > -dn_{0D}^L$ where $dn_{LD}^1 < 0$, (2) the RHS of (47) is increasing in $dn_{LD}^2$ since $n_{SD}^0 + dn_{0D}^S > 0$, and (3) $dn_{1D}^S = -dn_{0D}^S$, suffice for (47) to be decreasing in $A$ for $\{n_{SD}, n_{LD}\} = \{0.6, 0.1\}$ and $\{0.35, 0.35\}$ over the range of $A$ in which the respective unique equilibrium exists and is only increasing for a small initial range of $A$ for $\{n_{SD}, n_{LD}\} = \{0.1, 0.6\}$.
hold is that
\[ n_1^{SD} d_{n_0}^{LD} - (n_0^{LD} + d_{n_0}^{LD}) d_{n_0}^{SD} < -n_0^{SD} d_{n_0}^{LD} + (n_2^{LD} - d_{n_0}^{LD}) d_{n_0}^{SD} . \]
Collecting terms, the condition becomes
\[ -n^{SD} d_{n_0}^{LD} + (n^{LD} - n_1^{LD}) d_{n_0}^{SD} > 0 , \] which tends to hold for large \( |d_{n_0}^{LD}| \) relative to \( |d_{n_0}^{SD}| \). The large core dealers \( L_0 \)'s buy when they meet small peripheral dealers \( S_1 \)'s as well as other large core dealers \( L_2 \)'s, whereas the small peripheral dealers \( S_0 \)'s buy only when they meet large core dealers \( L_2 \)'s. No wonder \( |d_{n_0}^{LD}| \) does turn out to exceed \( |d_{n_0}^{SD}| \) in our quantitative analysis. In all, given the same ranking of the marginal benefits of inventory for the frictional inter-dealer market as for the competitive inter-dealer market, the implication that it is the small peripheral dealers that provide immediacy for the large core dealers largely remains.

### 4.3 Matching Opportunity

If each large dealer can hold up to two units in inventory and may possess up to two units of spare inventory capacity, perhaps a more natural assumption is that they can meet up to two investor-buyers and two investor-sellers in each period. We shall demonstrate below how the ranking of the marginal benefits of inventory in Proposition 1 can be left intact.

First, if a large dealer has up to two matching opportunities with investor-buyers and with investor-sellers, respectively, a reasonable matching technology should be such that

1. the probability that a large dealer meets at least one investor-buyer is weakly higher than the probability that a small dealer meets one investor-buyer,

2. the probability that a large dealer meets at least one investor-seller is weakly higher than the probability that a small dealer meets one investor-seller.

If, in addition, the matching technology exhibits diminishing returns in the sense that

3. the probability that a large dealer meets two investor-buyers is weakly lower than the probability that a small dealer meets one investor-buyer,

4. the probability that a large dealer meets two investor-sellers is weakly lower than the probability that a small dealer meets one investor-seller,

then the ranking in Proposition 1 remains.\(^{25}\)

\(^{24}\)That \( d_{n_0}^{LD} = -n_1^{LD} - n_0^{LD} \) and \( d_{n_0}^{SD} = -n_0^{SD} \) are due to respectively, \( n_0^{LD} + n_1^{LD} + n_2^{LD} = n^{LD} \) and \( n_0^{SD} + n_1^{SD} = n^{SD} \).

\(^{25}\)These assumptions are easily satisfied if the two matching outcomes for the large dealer are independent events. In this case, the probability that a large dealer meets at least one investor-buyer is

\[ 1 - (1 - \eta(\theta_{ID}))^2 = \eta(\theta_{ID})(2 - \eta(\theta_{ID})) > \eta(\theta_{ID}), \]

where the far-right term is the probability that the small dealer meets one investor-buyer. On the other hand, the probability that a large dealer meets as many as two investor-buyers is \( \eta(\theta_{ID})^2 < \eta(\theta_{ID}) \).
The arguments are as follows. First, consider the costs and benefits of acquiring the first unit of inventory in the inter-dealer market for the two types of dealers. Filling up the first unit of capacity in the inter-dealer market is costly to a small dealer as long as he shall meet one investor-seller in the next period but is costly to a large dealer only if he meets two investor-sellers (if a large dealer meets only one investor-seller, he still has capacity to buy) in the next period. By (4), the expected cost is higher for the small dealer. The expected benefits are higher for the large dealer — if (1) holds, the large dealer can sell the unit with weakly higher probability. Then, $V_{1}^{LD} - V_{0}^{LD} \geq V_{1}^{SD} - V_{0}^{SD}$ should follow. The inequality should be strict if either one of the relation in (1) or (4) is strict.

Next, consider the costs and benefits of utilizing the last unit of spare capacity for the two types of dealers. Exhausting one’s capacity is costly to a dealer, large or small, as long as the dealer shall meet one or more investor-seller in the next period. By (2), the expected cost is higher for the large dealer. A small dealer benefits from the additional unit of inventory if he meets one investor-buyer while a large dealer benefits only if he meets as many as two investor-buyers. If (3) holds, the expected benefit is higher for the small dealer. Then, $V_{1}^{SD} - V_{0}^{SD} \geq V_{2}^{LD} - V_{1}^{LD}$ should follow. The inequality should be strict if either one of the relation in (2) or (3) is strict.

5 Comparative Statics

Having shown how the major results of the model hold in more general settings, we now return to the basic model and study the model’s comparative statics.

5.1 Asset Supply

Market Tightness and Turnover In Section 3, we remarked that dealers should find it easier to buy but more difficult to sell in a market with more abundant asset supply. We state the formal results in the following proposition.

Proposition 4a (i) For $e < n^{LD}$, as $A$ increases from $B_L + e/\delta$ at which the Selling Equilibrium first holds, $\partial \theta_{DI} / \partial A = 0$ and $\partial \theta_{ID} / \partial A < 0$. Once $A$ reaches $B_M + e/\delta$ at which the Balanced Equilibrium begins to hold, $\partial \theta_{DI} / \partial A < 0$ and $\partial \theta_{ID} / \partial A < 0$. Finally, when $A$ rises up to and above $B_L + e/\delta$ at which the Buying Equilibrium holds, $\partial \theta_{DI} / \partial A < 0$ and $\partial \theta_{ID} / \partial A = 0$. In the transition from one equilibrium type to another, $\theta_{DI}$ and $\theta_{ID}$ are continuous. (ii) For $e \in \big[ n^{LD}, n^{LD} + n^{SD} \big]$ and that $A > B_M + e/\delta$ at which the Balanced Equilibrium holds, $\partial \theta_{DI} / \partial A < 0$ and $\partial \theta_{ID} / \partial A < 0$.

Proposition 4a implies that indeed, a given dealer-buyer meets an investor-seller at a (weakly) higher probability $\mu (\theta_{DI})$ while a given dealer-seller meets an investor-buyer at a (weakly) lower probability $\eta (\theta_{ID})$ as $A$ rises. Perhaps somewhat unexpected a priori is that $\theta_{DI}$ in the Selling Equilibrium and $\theta_{ID}$ in the Buying Equilibrium do not vary with $A$. In the Selling Equilibrium, given that all dealers enter the investor-dealer market with at least one unit of spare inventory capacity, all dealers are dealer-buyers in which case $n^{D}B$ remains fixed at $n^{D}$ throughout. To follow is the same $\theta_{DI}$ in the steady state for all admissible values of $A$. 34
for otherwise, the measure of assets bought by dealers ($n^D\mu (\theta_{DI})$) cannot remain equal to $e$. That $\theta_{ID}$ in the Buying Equilibrium does not vary with $A$ can be explained similarly.

**Inter-dealer Trading Volume** In the inter-dealer market, trades are driven by the infra-marginal buyers’ or sellers’ desire to rebalance inventories. The trading volume ($TV$) in the Selling, Balanced, and the Buying Equilibria are then given by, respectively,

$$TV = \begin{cases} m^S_1 + m^L_2, \\ m^L_0 & A \leq S + \frac{e}{\delta} \\ m^L_2 & A \geq S + \frac{e}{\delta}, \\ m^S_0 + m^L_0. \end{cases}$$

**Proposition 4b** The inter-dealer market trading volume changes non-monotonically with $A$, as depicted in the table below.

<table>
<thead>
<tr>
<th>Selling Equilibrium</th>
<th>Balanced Equilibrium</th>
<th>Buying Equilibrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \leq S + \frac{e}{\delta}$ small dealers sell</td>
<td>$A \geq S + \frac{e}{\delta}$ small dealers buy</td>
<td></td>
</tr>
<tr>
<td>$\frac{\partial TV}{\partial A} &gt; 0$</td>
<td>$\frac{\partial TV}{\partial A} &lt; 0$</td>
<td>$\frac{\partial TV}{\partial A} &gt; 0$</td>
</tr>
</tbody>
</table>

For $e < n^L_D$, the trading volume changes continuously as one equilibrium type changes to another, peaking at $TV = e \left(1 - \frac{e}{\delta^P}\right)$, when the Selling Equilibrium turns into the Balanced Equilibrium and when the Balanced Equilibrium turns into the Buying Equilibrium.

The Proposition essentially says that the inter-dealer market is most active when the asset supply is at a relatively low level but not at the lowest level or at a relatively high level but not at the highest level – a result due to the interactions of changing market tightness and optimal dealers’ inventories as brought about by variations in $A$. In particular, say, to begin with, the asset supply is at the lowest level for which very few dealers can buy from investors in a very tight market, whereby, trades in the inter-dealer market, driven by the successful dealer-buyers’ need to dispose the inventories they acquire from investors, can only be few and far between. As the asset supply rises, successful dealer-buyers become more numerous, whereby inter-dealer trades begin to pick up. When the asset supply, still at a relatively low level, reaches the point at which the Balanced Equilibrium takes hold, trading in the inter-dealer market changes into being driven by the successful dealer-sellers replenishing their inventories in the market. As $A$ increases further, there are fewer successful dealer-sellers and such needs to replenish inventories weaken. How inter-dealer trades are least numerous when $A$ is at the highest level, begin to pick up as $A$ first goes down, but then declines once $A$ falls down to a low enough level can be understood similarly.
Inter-dealer Trading Prices Proposition 4a shows that dealers find it easier to buy from and harder to sell to investors as $A$ increases. If dealers who have bought from investors tend to sell and dealers who have sold to investors tend to buy afterwards in the inter-dealer market, an increase in $A$ should be followed by an increase in supply and a decline in demand in the inter-dealer market and a concomitant decline in the inter-dealer market price. Besides, when one equilibrium type turns into another, the inter-dealer market price $p$ changes its anchor from one indifference condition to another. As such, a minute change in the asset supply can cause a catastrophic change in $p$.

Proposition 4c (i) For $e < n^{LD}$, as $A$ increases from $B_L + e/\delta$ at which the Selling Equilibrium first holds, $p$ is continuously decreasing in $A$. Once $A$ reaches $B_M + e/\delta$ at which the Balanced Equilibrium begins to hold, there will be a discrete fall in $p$, followed by further continuous decreases as $A$ increases further. Finally, when $A$ rises up to and above $B_L + e/\delta$ at which the Buying Equilibrium holds, there will be another discrete fall in $p$ all the way to zero. (ii) For $e \in \left[ n^{LD}, n^{LD} + \frac{w^{SD}}{2} \right]$ and that $A > B_M + e/\delta$ at which the Balanced Equilibrium begins to hold, $p$ is continuously decreasing in $A$ throughout.

5.2 Measure of Large Dealers $n^{LD}$

A major point of departure in the present model is that dealers are heterogeneous in inventory capacity. In the comparative statics exercises below, we hold constant the total measure of dealers and vary the measure of large dealers. To the extent that there is least diversity among dealers when all dealers are small dealers and when all dealers are large dealers, the comparative statics with respect to $n^{LD}$ we present below can also be interpreted as the impacts of the diversity of dealers in the dealer population on equilibrium outcomes.

Market Tightness and Turnover If more of the dealers are large dealers possessing a two-unit inventory capacity, there will be a greater overall inventory capacity among dealers. Then, first of all, there will tend to be more dealer-buyers. Furthermore, when more dealers are buying from investors, dealers’ overall inventory holding tends to increase as well, giving rise to there being more dealer-sellers.

Proposition 5a Holding fixed $n^D$, as $n^{LD}$ increases from the smallest admissible value for which the Balanced Equilibrium holds, $\partial \theta_{DI}/\partial n^{LD} > 0$ and $\partial \theta_{ID}/\partial n^{LD} < 0$. If and when the Balanced Equilibrium gives way to the Buying Equilibrium, $\partial \theta_{DI}/\partial n^{LD} > 0$ and $\partial \theta_{ID}/\partial n^{LD} = 0$. On the other hand, if and when the Balanced Equilibrium gives way to the Selling Equilibrium, $\partial \theta_{DI}/\partial n^{LD} = 0$ and $\partial \theta_{ID}/\partial n^{LD} = 0$. The two market tightness are continuous at the point at which the Balanced Equilibrium turns into either the Buying or the Selling Equilibrium.

The substantive implication of Proposition 5a is that the investor-sellers’ matching rate $\eta(\theta_{DI})$ and the investor-buyers’ matching rate $\mu(\theta_{ID})$ are both weakly increasing in $n^{LD}$. In this way, a market with relatively more large dealers functions strictly better at transferring units of the asset from low- to high-valuation investors, except when it is in the Selling Equilibrium. In the Selling Equilibrium, in which all dealers enter the investor-dealer market with
at least one unit spare inventory capacity, any and all dealers are dealer-buyers in the market no matter how the dealer population is divided between the two types. Then, there will be the same $\theta_{DI} = n^D_0/n^D_{ON} = n^D/n^D_{ON}$ for all $n^{LD}$ for which the Selling Equilibrium holds. Moreover, as every dealer buys and buys at the same probability $\mu(\theta_{DI})$ for any $n^{LD}$, there must also be the same inventory holding among dealers, to be followed by the same $n^S_D$ and therefore the same $\theta_{ID} = n^I_B/n^S_D$. All this means that as soon as the the Selling Equilibrium takes hold, the market’s efficacy at intermediating trade among investors reaches its constrained best and cannot undergo any improvement from further increases in dealers’ inventory capacities.

**Inter-dealer Trading Volume** While the Balanced Equilibrium holds, the trading volume in the inter-dealer market is given by $\max\{m_0^{LD}, m_2^{LD}\}$. For given trading probabilities, $L_0$s and $L_2$s entering the inter-dealer market should be more numerous if large dealers simply constitute a bigger fraction of the dealer population. Furthermore, given $n^{LD}$, if dealers sell to and buy from investors both at lower probabilities, there should be fewer large dealers remaining as $L_1$s and more becoming either $L_0$s or $L_2$s at the closing of the investor-dealer market. In all, in the Balanced Equilibrium, $TV$ should be increasing in $n^{LD}$.

While the Buying Equilibrium holds, the trading volume equals $m_0^{SD} + m_0^{LD}$. As small dealers are replaced one-for-one by large dealers, dealers leaving the investor-dealer market with an empty inventory should fall in numbers since the large (but not the small) dealers may replenish any inventories they sell to investors in the same period of time by buying from other investors. Similarly, while the Selling Equilibrium holds, the trading volume, $m_1^{SD} + m_2^{LD}$, should fall when small dealers are replaced one-for-one by large dealers, as large dealers may restore the inventory capacities they forego during which they buy from investors by selling to other investors in the meantime.

**Proposition 5b** Holding $n^D$ fixed, the trading volume in the inter-dealer market is increasing in $n^{LD}$ while the Balanced Equilibrium holds. Once the Buying or the Selling Equilibrium takes hold, the trading volume becomes decreasing in $n^{LD}$. $TV$ is continuous at where the Balanced Equilibrium turns into either the Buying or the Selling Equilibrium and reaches the highest level equal to $e(1 - \frac{\epsilon}{n^D})$ at the point of transition.

Proposition 5b shows that for any level of asset supply, the inter-dealer market is least active when there is little diversity in the dealer population with $n^{LD}$ either at the lowest or at the highest level. With more diversity as when $n^{LD}$ is at some intermediate level, the market becomes more active. Trading in the inter-dealer market in our model then is mainly driven by the heterogeneity of dealers, rather than by dealers possessing more than a unit of inventory capacity.

**Inter-dealer Trading Prices** For $n^{LD}$ below the thresholds in Corollary 1a for which the Balanced Equilibrium holds, $p = \beta(V_1^{SD} - V_0^{SD})$, where by (65) and (66) in the Appendix, respectively,

\begin{align}
V_0^{SD} &= W_0^{SD} + \mu(\theta_{DI}) \frac{p - \beta U_{ON}^L}{2}, \\
V_1^{SD} &= W_1^{SD} + \eta(\theta_{ID}) \frac{\beta(U_{ON}^H - U^B) - p}{2}.
\end{align}

37
The first equation says that an $S_0$ has asset value equal to the value of his outside option $W_0^{SD}$ plus the probability of trade times his share of the match surplus from trading with an investor-seller. The second equation, for the asset value of an $S_1$, can be interpreted similarly. By Proposition 5a, increases in $n^{LD}$, while the Balanced Equilibrium holds, cause $\theta_{ID}$ to go up and $\theta_{DI}$ to go down, from which dealers buy as well as sell at lower probabilities. To follow, both $V_0^{SD}$ and $V_1^{SD}$ tend to decline. The overall effect on $p = \beta (V_1^{SD} - V_0^{SD})$ then appears ambiguous. In one set of quantitative analysis that we undertake, we find that $p$ falls throughout, where $V_1^{SD}$ declines more than $V_0^{SD}$ does. This happens when the Balanced Equilibrium will turn into the Selling Equilibrium and also when the Balanced Equilibrium will turn into the Buying Equilibrium.\footnote{The numerical analyses assume $\eta(\theta) = 1 - e^{-\theta}$, $n^D = 1$, $e = 0.8$, $\delta = 0.1$, $\beta = 0.95$, and $A = 11$ for which the Balanced Equilibrium will turn into the Selling Equilibrium and $A = 12$ for which the Balanced Equilibrium will turn into the Buying Equilibrium. The equations for $p$, $\theta_{ID}$, and $\theta_{DI}$ in the Balanced Equilibrium are given by (80), (100), and (101), respectively, in the Appendix.} We suspect that this overall negative effect is due to the tendency that as investors’ matching rates go up amid the falling matching rates for dealers, high-valuation non-owners should stand to gain more from the faster acquisition of the asset than low-valuation owners from the faster disposition of the asset in the steady state.\footnote{An investor-buyer benefits not just from there being a higher buying rate, but also from there being a higher selling rate since he can look forward to disposing of the unit he will hold later on faster if and when he suffers the liquidity shock. On the other hand, there is not any channel from which an investor-seller may benefit from a higher buying rate. In the steady state, then, $U^B$ tends to increase more than $U^{ON}$ does except perhaps when future payoffs are heavily discounted and when the investor-seller matching rate rises significantly more than the investor-buyer matching rate does.} Any larger increase in $U^B$ than in $U^{ON}$, according to (49) and (50), should exert a negative effect on $V_1^{SD} - V_0^{SD}$ – an effect that apparently is of overriding importance in our quantitative analysis.

**Proposition 5c** Holding fixed $n^D$, as $n^{LD}$ rises, if and when the Balanced Equilibrium gives way to the Buying Equilibrium, $p$ falls by a discrete amount down to zero; if and when the Balanced Equilibrium gives way to the Selling Equilibrium, $p$ jumps up by a discrete amount and stays at the same level for all $n^{LD}$. The inter-dealer market price may vary with $n^{LD}$ to the extent that either one or both of the market tightness vary in response to the change in $n^{LD}$. By Proposition 5a, as soon as the Selling Equilibrium takes hold, the two market tightness reach their respective maximum and minimum values. In the meantime, by Proposition 5c, $p$ attains its highest possible value in equilibrium when it becomes anchored at $p = \beta (V_1^{LD} - V_0^{LD})$ – the largest marginal benefit of inventory. A corollary of the Proposition and our previous quantitative analysis is that $p$ can be non-monotonic with respect to increases $n^{LD}$, first decreasing while the Balanced Equilibrium holds, reaching the minimum at the transition to the Selling Equilibrium, and then going up by a discrete amount thereafter.\footnote{While the Balanced Equilibrium holds, $V_1^{LD} - V_0^{LD}$, like $V_1^{SD} - V_0^{SD}$, can also be decreasing in $n^{LD}$. If $p$ were anchored at $\beta (V_1^{LD} - V_0^{LD})$ for all $n^{LD}$, it should not increase at the transition from the Balanced to the Selling Equilibria at which point the two market tightness reach their respective maximum and minimum. But $p$ is not anchored at $\beta (V_1^{SD} - V_0^{SD})$ but at the lower $\beta (V_1^{SD} - V_0^{SD})$ while the Balanced Equilibrium holds. The possible non-monotonicity arises from the change in the anchor of $p$ at the transition.}
Investor-Dealer Market Prices  For brevity, in Propositions 4c and 5c, we have not ex-
tended the analysis to also checking how prices in the investor-dealer market may vary with $A$
and $n^{LD}$. In Propositions A1 and A2 and the ensuing discussions in Appendix 7.2, we show
that the dealers’ ask and bid prices in the market do turn out to vary with $A$ and $n^{LD}$ in the
just the same ways that the inter-dealer market price does.

6 Conclusion

In this paper, by means of a tractable random search model, we study inter-dealer trades among
heterogeneous dealers in OTC markets motivated by inventory risk concerns. We depart from
earlier such models by all assuming that all traders are risk neutral. Even so, the dealers benefit
from trading among one another to eliminate the risks of carrying an insufficient inventory and
an insufficient spare inventory capacity for their trading needs with investors. The inter-
dealer trading network that emerges endogenously resembles a core-periphery structure, with
large dealers in the center, trading among themselves and with small dealers, who are in the
periphery, trading with large core dealers only. The large core dealers each hold weakly more
units of inventory in equilibrium than a small peripheral dealer does. These features match a
number of the stylized facts documented in the literature.

The model yields a rich set of testable implications for future research. First and foremost,
our analysis shows that the apparently obvious notion that large core dealers should provide
inventory for small peripheral dealers insofar as the latter have larger inventory capacities only
holds up when inventory is relatively abundant. But in this case, small dealers should need
the inventory the least. In contrast, in our model, it is the small peripheral dealers who trade
to provide immediacy for the large core dealers, selling to large core dealers when they need
inventory the most and buying from large core dealers when they need liquidity the most.

We show in Appendix 7.3 that such features of equilibrium inter-dealer trades actually help
attain constrained efficiency. In the planning optimum, inventories are allocated to dealers to
enable high-valuation investors to acquire the asset most rapidly and to enable units of the
asset to be transferred from low-valuation investors to dealers the quickest, thereby facilitat-
ing the eventual sales to the high-valuation investors. In the competitive inter-dealer market,
inventories and spare capacities are allocated to dealers who value them the most – the very
dealers who have the best use of them for trading with investors. Perhaps not surprisingly, the
equilibrium allocations coincide with the constrained optimum allocations. More interestingly,
our analysis suggests that for efficiency, the peripheral small dealers should indeed trade to
provide immediacy for the large core dealers.
7 Appendix

7.1 The Frictional Inter-Dealer Market Model

The model is set in continuous time in which a dealer meets another randomly selected dealers at the rate $\alpha$ per time unit. All other notations have the same meanings as for the main model.

7.1.1 Value Functions

To define the value functions, we rule out a priori any exchanges between two dealers that merely result in the two dealers concerned switching states as such exchanges cannot give rise to a positive surplus.

**Small dealers** An $S_0$, who can only buy, meets an investor-seller at the rate $\mu(\theta_{DI})$ and another dealer at the rate $\alpha$. Among all dealers that the $S_0$ may meet, there can be a potentially profitable exchange only if the counterparty is an $L_1$ or an $L_2$. By Lemma 5, all investor-dealer trades yield non-negative surpluses. Then,

$$rV^0_{SD} = \mu(\theta_{DI})\left(V^SD_{1} - V^SD_0 - p_{S_0,L_3}\right) + \alpha\left\{\frac{n_{LD}}{n_D} \max \left\{ -p_{S_0,L_1} + V^SD_1 - V^SD_0, 0 \right\} \right.$$  

$$+ \frac{n_{LD}^2}{n_D} \max \left\{ -p_{S_0,L_2} + V^SD_1 - V^SD_0, 0 \right\} \right. .$$

An $S_1$, who can only sell, meets an investor-buyer at the rate $\eta(\theta_{ID})$. The $S_1$ may also sell to an $L_0$ or an $L_1$. Then,

$$rV^1_{SD} = \eta(\theta_{ID})\left(p_{I_B,S_1} + V^SD_0 - V^SD_1\right) + \alpha\left\{\frac{n_{LD}}{n_D} \max \left\{ p_{I_B,S_1} + V^SD_0 - V^SD_1, 0 \right\} \right.$$  

$$+ \frac{n_{LD}^2}{n_D} \max \left\{ p_{I_1,S_1} + V^SD_0 - V^SD_1, 0 \right\} \right. .$$

**Large dealers** An $L_0$ may buy from an investor-seller, an $S_1$, or an $L_2$. Then,

$$rV^0_{LD} = \mu(\theta_{DI})\left(V^LD_1 - V^LD_0 - p_{L_0,L_3}\right) + \alpha\left\{\frac{n_{SD}}{n_D} \max \left\{ -p_{L_0,S_1} + V^LD_1 - V^LD_0, 0 \right\} \right.$$  

$$+ \frac{n_{SD}^2}{n_D} \max \left\{ -p_{L_0,L_2} + V^LD_1 - V^LD_0, 0 \right\} \right. .$$

An $L_1$ may buy from an investor-seller and sell to an investor-buyer. Among dealers, he may sell to an $S_0$, buy from an $S_1$, and either buy from or sell to another $L_1$. Then,

$$rV^1_{LD} = \mu(\theta_{DI})\left(V^2_{LD} - V^1_{LD} - p_{L_1,L_3}\right) + \eta(\theta_{ID})\left(p_{I_B,L_1} + V^1_{LD} - V^1_{LD}\right)$$  

$$+ \alpha\left\{\frac{n_{SD}}{n_D} \max \left\{ p_{S_0,L_1} + V^LD_0 - V^LD_1, 0 \right\} + \frac{n_{SD}^2}{n_D} \max \left\{ -p_{L_1,S_1} + V^2_{LD} - V^1_{LD}, 0 \right\} \right.$$  

$$+ \frac{n_{LD}}{n_D} \max \left\{ -p_{L_1,L_1} + V^2_{LD} - V^1_{LD}, p_{L_1,L_1} + V^0_{LD} - V^1_{LD}, 0 \right\} \right. .$$
An $L_2$, who may only sell, meets an investor-buyer at the rate $\eta(\theta_{ID})$. Among dealers, he may sell to an $S_0$ or an $L_0$. Then,

\[ rV_{LD}^2 = \eta(\theta_{ID}) \left( p_{LB,L2} + V_{1LD} - V_{2LD} \right) + \alpha \left\{ n_{SD}^{LD} \max \left\{ p_{S_0,L2} + V_{1LD} - V_{2LD}, 0 \right\} + \frac{n_{0}^{LD}}{n_{I}^{LD}} \max \left\{ p_{L_0,L2} + V_{1LD} - V_{2LD}, 0 \right\} \right\} . \]

**Investors** An investor-buyer may buy from an $S_1$, an $L_1$, or an $L_2$. Then,

\[ rU^B = \mu(\theta_{ID}) \left( U_{HON}^B - U^B - \frac{n_{SD}^{LD}}{n_{S}^{LD}} p_{I_B,S_1} - \frac{n_{1}^{LD}}{n_{S}^{LD}} p_{I_B,L_1} - \frac{n_{2}^{LD}}{n_{S}^{LD}} p_{I_B,L_2} \right); \]

where

\[ rU_{HON}^B = \nu + \delta \left( U_{LON}^B - U_{HON}^B \right). \]

An investor-seller may sell to an $S_0$, an $L_0$, or an $L_1$. Then,

\[ rU_{LON}^B = \eta(\theta_{DI}) \left( \frac{n_{SD}^{LD}}{n_{B}^{LD}} p_{S_0,I_S} + \frac{n_{0}^{LD}}{n_{B}^{LD}} p_{I_0,I_S} + \frac{n_{1}^{LD}}{n_{B}^{LD}} p_{I_1,I_S} - U_{LON}^B \right). \]

### 7.1.2 Equilibrium Conditions

First, where (29)-(31) hold for both the competitive and the frictional inter-dealer market models, the same equations for $n_{HON}^I$, $n_{LON}^I$, and $n_{I}^I$ derived in the proof of Proposition 2 hold. By (26) then,

\[ n_{1}^{SD} = A - \frac{\epsilon}{\delta} - \frac{\epsilon}{\eta(\theta_{DI})} - n_{1}^{LD} - 2n_{2}^{LD}, \]  

(51)

from which it follows that

\[ n_{1}^{LD} + n_{1}^{SD} + n_{2}^{LD} = A - \frac{\epsilon}{\delta} - \frac{\epsilon}{\eta(\theta_{DI})} - n_{2}^{LD}. \]

Substitute the equation into (27) to yield

\[ n_{2}^{LD} = A - \frac{\epsilon}{\delta} - \frac{\epsilon}{\eta(\theta_{DI})} - \frac{\epsilon}{\eta(\theta_{ID})}. \]  

(52)

Next, by (24), (25), (51), and (52),

\[ n_{0}^{SD} + n_{0}^{LD} + n_{1}^{LD} = n_{D} - \frac{\epsilon}{\eta(\theta_{ID})} + n_{1}^{LD}. \]

Substitute the equation into (28) to yield

\[ n_{1}^{LD} = \frac{\epsilon}{\mu(\theta_{DI})} + \frac{\epsilon}{\eta(\theta_{ID})} - n_{D}. \]  

(53)

It is then straightforward to derive the following,

\[ n_{0}^{SD} = A - \frac{\epsilon}{\delta} - n_{1}^{LD} - \frac{\epsilon}{\eta(\theta_{DI})} - \frac{\epsilon}{\eta(\theta_{ID})} + \frac{\epsilon}{\mu(\theta_{DI})}. \]  

(54)
\[ n_1^{SD} = n^D - A + \frac{e}{\delta} + \frac{e}{\eta(\theta_{DI})} + \frac{e}{\eta(\theta_{ID})} - \frac{e}{\mu(\theta_{DI})}, \]  
\[ n_0^{LD} = n^{LD} - A + \frac{e}{\delta} - \frac{e}{\mu(\theta_{DI})} - \frac{e}{\eta(\theta_{DI})}. \]

Any \( \{\theta_{ID}, \theta_{DI}\} \) pair satisfying (43) and (44) with the measures of dealers given by (51)-(56), where the resulting \( n_0^{SD}, n_1^{SD} \in [0, n^{SD}] \) and \( n_0^{LD}, n_1^{LD}, n_2^{LD} \in [0, n^{LD}] \), is a steady-state equilibrium.

7.2 Dealers’ Bid and Ask Prices

**Lemma A1** In all three types of equilibrium, the dealers’ bid price; i.e., the price at which investors sell to dealers is given by

\[ p_{IS} = \frac{1 - \beta + \beta \eta(\theta_{DI})}{2(1 - \beta) + \beta \eta(\theta_{DI})} p, \]

whereas the dealers’ ask prices; i.e., the prices at which investors buy from dealers in the Selling, Balanced, and Buying Equilibria are given by, respectively,

\[ p_{IB} = \left(1 + \frac{1 - \beta}{\beta \eta(\theta_{ID})}\right) p, \]

\[ p_{IB} = \left(1 + \frac{1 - \beta + \beta \mu(\theta_{DI})}{\beta \eta(\theta_{ID})(4(1 - \beta) + 2\beta \eta(\theta_{DI}))} \right) p, \]

\[ p_{IB} = \frac{\beta (1 - \beta) v}{(1 - \beta + \beta \delta)(2(1 - \beta) + \mu(\theta_{ID}) \beta)}. \]

**Proposition A1** For \( e \leq n^{LD} \), as \( A \) increases from \( B_L + e/\delta \) at which the Selling Equilibrium first holds, \( p_{IS} \) and \( p_{IB} \) are continuously decreasing in \( A \). Once \( A \) reaches \( B_M + e/\delta \) at which the Balanced Equilibrium begins to hold, there will be discrete falls in the two prices. While the Balanced Equilibrium holds, \( p_{IS} \) is continuously decreasing in \( A \). And then finally, when \( A \) rises up to and above \( B_L + e/\delta \), at which the Buying Equilibrium holds, there will be further discrete falls in the two prices – \( p_{IS} \) all the way to zero and \( p_{IB} \) to some positive value. Thereafter, the two prices do not vary with \( A \) any longer. For \( e \in \left[n^{SD}, n^{LD} + n^{SD}/2\right] \) and that \( A > B_M + e/\delta \) at which the Balanced Equilibrium begins to hold, \( p_{IS} \) is likewise continuously decreasing in \( A \).

The Proposition leaves out how \( p_{IB} \) may vary with \( A \) in the Balanced Equilibrium as it does not seem possible to sign \( \partial p_{IB}/\partial A \) in said equilibrium. Our numerical analyses do reveal, however, that \( p_{IB} \) does decline in \( A \) while the Balanced Equilibrium holds,\(^\text{29}\) just as \( p \) and \( p_{IS} \) do.

\(^{29}\) Under the same parameter configurations as for the numerical analyses preceding Proposition 4c, except that \( n_{LD} \) is fixed at 0.846.
For the smallest admissible $n^{LD}$, the market starts off in a Balanced Equilibrium, in which $p$, as our numerical analyses in the main text indicate, tends to decline with increases in $n^{LD}$. In the same numerical analyses, we find that $p_{IS}$ and $p_{IB}$ follow the same tendency.

**Proposition A2**  
Holding fixed $n^{D}$, as $n^{LD}$ rises, if and when the Balanced Equilibrium gives way to the Buying Equilibrium, both $p_{IS}$ and $p_{IB}$ fall by some discrete amount – $p_{IS}$ to zero and $p_{IB}$ to some positive value; if and when the Balanced Equilibrium gives way to the Selling Equilibrium, both $p_{IS}$ and $p_{IB}$ jump up by some discrete amount. In either the Buying or the Selling Equilibrium, the two prices do not vary with $n^{LD}$.

### 7.3 Efficient Decentralized Market Trades

A social planner maximizes the discounted flow payoffs for investors over time from the ownership of the asset given by,

$$W = \max \left\{ \sum_{t=0}^{\infty} \beta^t n^{ON}_H(t) \nu \right\},$$  
(61)

subject to the same search and matching frictions that agents in the model face.

A priori, the equilibrium trades in the frictional investor-dealer market are constrained efficient where any trades with a positive surplus, but only such trades, will take place with the terms of trade in the bilateral meetings reached via Nash Bargaining. Specifically, any investor-buyer and dealer-seller trade is efficient with the former, but not the latter, deriving the flow payoff $\nu$ in holding a unit of the asset. But then a dealer-seller becomes a dealer-seller in the first place only by acquiring the asset from an investor-seller. Then, any and all trades between an investor-seller and a dealer-buyer are also efficient.

This means that it suffices for us to ask how the planner may wish to allocate units of inventory among the dealers in each period after the investor-dealer trades are completed and whether the allocation coincides with the allocation that falls out from the inter-dealer market in equilibrium.

**Lemma A2**  
In the steady state of the planner’s solution, units of inventory not held by investors are allocated to dealers to maximize the measures of dealer-sellers (dealers who hold inventory) and dealer-buyers (dealers who possess spare capacity). To maximize the measure of dealer-sellers, first allocate one unit each to either small or large dealers, and then allocate any remaining inventory to the large dealers. To maximize the measure of dealer-buyers, first allocate one unit each to large dealers, and then allocate any remaining inventory to either large or small dealers. The two objectives are then attained simultaneously by allocating inventory in the following order: (1) one unit each to large dealers; (2) if there remains any inventory, then one unit each to small dealers; (3) if there remains any inventory, one more unit each to large dealers.

**Proposition A3**  
In the steady state, the allocations from the decentralized market trades coincide with the planning optimum.
7.4 Proofs of Lemmas and Propositions

**Proof of Lemma 1** Notice that

\[ W_1^{SD} = W_0^{SD} + p, \]  
\[ W_0^{LD} = W_1^{LD} - p, \]  
\[ W_1^{LD} = W_1^{LD} + p. \]

The lemma then follows from (14)-(19).

**Proof of Proposition 1** Substitute (21) and (23) and (62)-(64) into the value functions (6) and (7) and (11)-(13),

\[ V_0^{SD} = W_0^{SD} + \frac{\mu (\theta_{DI})}{2} (p - \beta U_L^{ON}), \]  
\[ V_1^{SD} = W_0^{SD} + \left(1 - \frac{\eta (\theta_{ID})}{2}\right) p + \frac{\eta (\theta_{ID})}{2} \beta (U_H^{ON} - U_B), \]  
\[ V_0^{LD} = W_1^{LD} - \left(1 - \frac{\mu (\theta_{DI})}{2}\right) p - \frac{\mu (\theta_{DI})}{2} \beta U_L^{ON}, \]  
\[ V_1^{LD} = W_1^{LD} + \frac{\mu (\theta_{DI}) - \eta (\theta_{ID})}{2} p - \frac{\mu (\theta_{DI})}{2} \beta U_L^{ON} + \frac{\eta (\theta_{ID})}{2} \beta (U_H^{ON} - U_B), \]  
\[ V_2^{LD} = W_1^{LD} + \left(1 - \frac{\eta (\theta_{ID})}{2}\right) p + \frac{\eta (\theta_{ID})}{2} \beta (U_H^{ON} - U_B). \]

We can then calculate

\[ (V_1^{LD} - V_0^{LD}) - (V_1^{SD} - V_0^{SD}) = \frac{\mu (\theta_{DI})}{2} (p - \beta U_L^{ON}), \]  
\[ (V_1^{SD} - V_0^{SD}) - (V_2^{LD} - V_1^{LD}) = \frac{\eta (\theta_{ID})}{2} (\beta (U_H^{ON} - U_B) - p). \]

Notice that the terms inside the brackets in (70) and (71) denote, respectively, the surpluses of trade between an investor-seller and any dealer-buyer and between an investor-buyer and any dealer-seller in (20) and (22). If either of the two is negative, there cannot be any trade in equilibrium between investors and dealers in the steady state.

**Proof of Lemma 2** Substitute (21) into (3) and rearrange,

\[ U_L^{ON} = \frac{\eta (\theta_{DI})}{2} p \]  
\[ U_H^{ON} = \frac{(1 - \beta + \beta \eta (\theta_{DI})^2 p) \left(1 - \beta + \frac{\eta (\theta_{DI})}{2} \beta \right)}{(1 - \beta + \beta \delta) \left(1 - \beta + \frac{\eta (\theta_{DI})}{2} \beta \right)} \]
Substitute (23) into (1) and rearrange,

\[ U^B = \frac{\mu(\theta_{TD})}{1 - \beta + \beta \frac{\eta(\theta_{PD})}{2}} (\beta U^N_H - p). \]  

(74)

Substituting from (73),

\[ U^B = \mu(\theta_{ID}) \frac{\beta}{2} \left( 1 - \beta + \frac{\eta(\theta_{PD})}{2} \right) v - \left( 1 - \beta \right) \left( 1 - \beta + \beta \delta + \frac{\eta(\theta_{PD})}{2} \right) p \]

\[ \frac{1 - \beta + \beta \delta \left( 1 - \beta + \beta \frac{\eta(\theta_{PD})}{2} \right) \left( 1 - \beta + \frac{\eta(\theta_{PD})}{2} \right)}{1 - \beta + \beta \delta \left( 1 - \beta + \beta \frac{\eta(\theta_{PD})}{2} \right) \left( 1 - \beta + \frac{\eta(\theta_{PD})}{2} \right)}. \]  

(75)

Then, by (73) and (75),

\[ U^N_H - U^B = \frac{\left( \mu(\theta_{TD}) \right)_{(1-\beta+\beta\delta)} \left( 1 - \beta + \beta \delta \left( 1 - \beta + \beta \frac{\eta(\theta_{PD})}{2} \right) \left( 1 - \beta + \frac{\eta(\theta_{PD})}{2} \right) \right) v + \left( 1 - \beta + \beta \delta \left( 1 - \beta + \beta \frac{\eta(\theta_{PD})}{2} \right) \left( 1 - \beta + \frac{\eta(\theta_{PD})}{2} \right) \right) p}{1 - \beta + \beta \delta \left( 1 - \beta + \beta \frac{\eta(\theta_{PD})}{2} \right) \left( 1 - \beta + \frac{\eta(\theta_{PD})}{2} \right)}. \]  

(76)

Set \( p = \beta \left( V^L_{1D} - V^L_{0D} \right) \) and by (67) and (68),

\[ p = \frac{\beta \eta(\theta_{ID})}{1 - \beta + \beta \frac{\eta(\theta_{PD})}{2}} (U^N_H - U^B). \]  

(77)

Then use (76) to obtain

\[ p = \frac{\beta^2 \eta(\theta_{ID})}{2} \left( 1 - \beta + \frac{\eta(\theta_{PD})}{2} \right) v \]

\[ \left( 1 - \beta + \beta \delta \left( 1 - \beta + \beta \frac{\eta(\theta_{PD})}{2} \right) \left( 1 - \beta + \frac{\eta(\theta_{PD})}{2} \right) \right) + \beta^2 \eta(\theta_{ID}) \left( 1 - \beta + \beta \delta \left( 1 - \beta + \beta \frac{\eta(\theta_{PD})}{2} \right) \left( 1 - \beta + \frac{\eta(\theta_{PD})}{2} \right) \right) \left( 1 - \beta + \beta \delta \left( 1 - \beta + \beta \frac{\eta(\theta_{PD})}{2} \right) \left( 1 - \beta + \frac{\eta(\theta_{PD})}{2} \right) \right). \]  

(78)

Given the positivity of \( p \) in (78) and by (72) and (77),

\[ 0 < \beta U^N_H < p < \beta (U^N_H - U^B). \]

Next, set \( p = \beta \left( V^S_{1D} - V^S_{0D} \right) \) and by (65) and (66),

\[ p = \eta(\theta_{ID}) \beta^2 \left( U^N_H - U^B \right) + \frac{\mu(\theta_{PD})}{2} \beta^2 \frac{U^N_H}{1 - \beta + \beta \frac{\eta(\theta_{ID})}{2} + \frac{\mu(\theta_{PD})}{2}}. \]  

(79)

Then use (72) and (76) to obtain

\[ p = \frac{\eta(\theta_{ID}) \beta^2 \left( 1 - \beta + \frac{\eta(\theta_{PD})}{2} \right) v}{\left( 1 - \beta + \beta \frac{\eta(\theta_{PD})}{2} \right) \left( 1 - \beta + \beta \delta \left( 1 - \beta + \beta \frac{\eta(\theta_{PD})}{2} \right) \left( 1 - \beta + \frac{\eta(\theta_{PD})}{2} \right) \right) \left( 1 - \beta + \beta \delta \left( 1 - \beta + \beta \frac{\eta(\theta_{PD})}{2} \right) \left( 1 - \beta + \frac{\eta(\theta_{PD})}{2} \right) \right)}. \]  

(80)

Given the positivity of \( p \) in (80) and by (72) and (79),

\[ 0 < \beta U^N_H < p < \beta (U^N_H - U^B). \]
Finally, set \( p = \beta (V_2^{LD} - V_1^{LD}) \) and by (68) and (69),
\[
p = \frac{\beta \mu(\theta_{ID})}{1 - \beta + \beta \mu(\theta_{ID})} U_L^{ON}.
\] (81)

With (72),
\[
p = \beta U_L^{ON} = 0.
\]

Next, by (76),
\[
U_H^{ON} - U_B = \frac{(1-\beta) \nu}{(1-\beta+\beta \delta) \left(1-\beta + \mu(\theta_{ID})\right)} > 0.
\] (82)

Thus,
\[
0 = \beta U_L^{ON} = p < \beta \left( U_H^{ON} - U_B \right).
\]

**Proof of Proposition 2** Before proceeding to prove the Proposition, it is useful to establish the following.

**Remark 1** For \( x \leq 1 \), \( x < \eta^{-1}(x) \).
Proof. Given \( \mu(x) = \eta(x)/x \) and that \( \mu(x) < 1, \eta(x) < x \). And then for \( x \leq 1 \), the last condition implies \( x < \eta^{-1}(x) \).

**Remark 2** For \( x \geq 1 \), \( x > \mu^{-1} \left( \frac{1}{2} \right) \).
Proof. Given that \( \eta(x) = x \mu(x) \) and that \( \eta(x) < 1, \mu(x) < \frac{1}{2} \). And then for \( x \geq 1 \), the last condition implies \( x > \mu^{-1} \left( \frac{1}{2} \right) \).

**Remark 3** For \( e \leq n \), \( \frac{n}{\mu^{-1}(\frac{e}{n})} \) is decreasing in \( n \).
Proof. By differentiation.

To begin proving the Proposition, we start with manipulating (29)-(31) to obtain,
\[
n_H^{ON} = \frac{e}{\delta},
\] (83)
\[
n_L^{ON} = \frac{e}{\eta(\theta_{DI})},
\] (84)
\[
n_B^{L} = \frac{e}{\mu(\theta_{ID})}.
\] (85)

**Selling Equilibrium** In the Selling Equilibrium, \( n_2^{LD} = n_1^{SD} = 0 \) and \( n_0^{SD} = n^{SD} \). Then, together with (84) and (85), the two market tightness equations, (27) and (28), specialize to, respectively,
\[
\eta(\theta_{ID}) = \frac{e}{n_1^{LD}},
\] (86)
\[
\mu(\theta_{DI}) = \frac{e}{n^{ID}}.
\] (87)
By (26), (83), (84), and that \( n_{1D}^S = n_{2D}^L = 0 \) in the Selling Equilibrium,

\[
n_{1D}^L = A - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})}. \tag{88}
\]

Substitute the equation into (86) and rearrange,

\[
\eta(\theta_{ID}) = \frac{\eta(\theta_{DI})e}{(A - \frac{e}{\delta})\eta(\theta_{DI}) - e}. \tag{89}
\]

Once \( \theta_{DI} \) is known from (87), the above uniquely gives \( \theta_{ID} \). For \( \theta_{DI} \) from (87) to be a valid equilibrium, it has to be such that the resulting: (a) \( \eta(\theta_{ID}) \in (0,1) \), as given by (89) and (b) \( n_{1D}^L \in (e, n_{LD}) \), as given by (88). For (b) to be satisfied, \( e < n_{LD}^L \) must hold.

By (89), for \( \eta(\theta_{ID}) \in (0,1) \),

\[
\frac{e}{\eta(\theta_{DI})} < A - \frac{e}{\delta} - e. \tag{RS.1}
\]

Substituting from (87) and rearranging, (RS.1) holds if

\[
A - \frac{e}{\delta} > e + \frac{n_{LD}^L}{\mu^{-1}(\frac{e}{n_{LD}})} = B_S. \tag{AS.1}
\]

Given that \( \eta(\theta_{ID}) < 1 \), \( n_{1D}^L > e \) holds for sure. By (88), for \( n_{1D}^L \leq n_{LD}^L \),

\[
\frac{e}{\eta(\theta_{DI})} \geq A - \frac{e}{\delta} - n_{LD}^L. \tag{RS.2}
\]

Given \( \theta_{DI} \) from (87), the above becomes,

\[
A - \frac{e}{\delta} \leq n_{LD}^L + \frac{n_{LD}^L}{\mu^{-1}(\frac{e}{n_{LD}})} = B_M. \tag{AS.2}
\]

Notice that for (RS.1) and (RS.2) to be satisfied at the same time, it has to be such that \( e < n_{LD}^L \), in which case (87) and the two conditions (AS.1) and (AS.2) are guaranteed to be well-defined.

Substituting (33), (34), and (36) into the inter-dealer market equilibrium condition (37),

\[
(n_{SD} + n_{1D}^L) \mu(\theta_{DI}) \leq n_{0D}^L (1 - \mu(\theta_{DI})) + n_{1D}^L \eta(\theta_{ID}).
\]

And then with (86)-(89), the condition can be shown to simplify to (RS.2).

To sum up, the Selling Equilibrium holds if and only if (AS.1) and (AS.2) hold; i.e., \( A - \frac{e}{\delta} \leq \in (B_S, B_M) \), in addition to \( e < n_{LD}^L \).

**Buying Equilibrium** In the Buying Equilibrium, \( n_{0D}^L = n_{0D}^S = 0 \) and \( n_{1D}^S = n_{SD}^L \). Then, together with (84) and (85), the two market tightness equations, (27) and (28), specialize to, respectively,

\[
\eta(\theta_{ID}) = \frac{e}{n_{LD}^L}, \tag{91}
\]
\[ \mu(\theta_{DI}) = \frac{e}{n_{LD}}. \]  

(92)

By (26), (83), (84), and that \( n_0^{SD} = n_0^{LD} = 0 \) in the Buying Equilibrium,
\[ n_1^{LD} = n^{SD} + 2n^{LD} - A + \frac{e}{\delta} + \frac{e}{\eta(\theta_{DI})}. \]  

(93)

Substitute the equation into (92),
\[ e(\theta_{DI} - 1) - \eta(\theta_{DI}) \left( n^{SD} + 2n^{LD} - A + \frac{e}{\delta} \right) = 0, \]  

(94)

which is an equation in \( \theta_{DI} \) alone. It is straightforward to verify that there is a unique positive solution of \( \theta_{DI} \) to the equation, and that the LHS is positive (negative) for \( \theta_{DI} \) above (below) the solution of the equation. For the solution to be a valid equilibrium, it has to be such that the resulting \( n_1^{LD} \in (e, n^{LD}) \), as given by (93). Then \( e < n^{LD} \) must be satisfied.

Rearranging (93), \( n_1^{LD} \leq n^{LD} \) if and only if
\[ \frac{e}{\eta(\theta_{DI})} \leq A - \frac{e}{\delta} - n^{D}. \]  

(RB.1)

A necessary condition for the equation to hold is that
\[ A - \frac{e}{\delta} - n^{D} \geq e. \]  

(AB.1.a)

Then, (RB.1) holds if the LHS of (94) is non-positive when evaluated at \( \theta_{DI} = \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - n^{D}} \right) \).

Where \( e < n^{LD} \), which is necessary for the RHS of (92) and also guarantees the RHS of (91) to be bounded below one, the condition reads
\[ A - \frac{e}{\delta} - n^{D} - 2n^{LD} \leq 0, \]  

(AB.1.b)

which subsumes (AB.1.a).

Rearranging (93), \( n_1^{LD} > e \) if and only if
\[ A - \frac{e}{\delta} + e - n^{SD} - 2n^{LD} < \frac{e}{\eta(\theta_{DI})} \]  

(RB.2)

Hence, if
\[ A - \frac{e}{\delta} - n^{SD} - 2n^{LD} \leq 0, \]
then (RB.2) holds for sure. Otherwise, the condition holds if the LHS of (94) is positive when evaluated at \( \theta_{DI} = \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} + e - n^{SD} - 2n^{LD}} \right) \); i.e.,
\[ \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} + e - n^{SD} - 2n^{LD}} \right) > \frac{e}{A - \frac{e}{\delta} + e - n^{SD} - 2n^{LD}}. \]

But the condition is guaranteed to hold by Remark 1.

Substituting (32), (34), and (36) into the inter-dealer market equilibrium condition (41),
\[ (n^{SD} + n_1^{LD}) \eta(\theta_{ID}) \leq n_1^{LD} \mu(\theta_{DI}) + n_2^{LD} (1 - \eta(\theta_{ID})). \]

Then, by (91)-(94), the condition can be shown to simplify to (RB.1).

To sum up, the Buying Equilibrium holds if and only if (AB.1b) holds; i.e., \( A - \frac{e}{\delta} \geq B_L \), in addition to \( e < n^{LD} \).
Balanced Equilibrium  In the Balanced Equilibrium,  \( n_{0}^{LD} = n_{2}^{LD} = 0 \) and  \( n_{1}^{LD} = n^{LD} \). Then, together with (84) and (85), the two market tightness equations, (27) and (28), specialize to, respectively,

\[
\eta(\theta_{ID}) = \frac{e}{n_{1}^{SD} + n^{LD}}, \tag{96}
\]

\[
\mu(\theta_{DI}) = \frac{e}{n_{0}^{SD} + n^{LD}}. \tag{97}
\]

By (26), (83), and (84), and that  \( n_{0}^{LD} = n_{2}^{LD} = 0 \) in the Balanced Equilibrium,

\[
n_{1}^{SD} = A - n^{LD} - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})}, \tag{98}
\]

and therefore

\[
n_{0}^{SD} = n^{D} - A + \frac{e}{\delta} + \frac{e}{\eta(\theta_{DI})}. \tag{99}
\]

Substituting (98) and (99) into (96) and (97), respectively, and rearranging,

\[
\eta(\theta_{ID}) = \frac{\eta(\theta_{DI}) e}{(A - e/\delta) \eta(\theta_{DI}) - e}, \tag{100}
\]

\[
e(\theta_{DI} - 1) - \eta(\theta_{DI}) \left( n^{D} + n^{LD} - A + \frac{e}{\delta} \right) = 0, \tag{101}
\]

which are respectively the same equations that give \( \theta_{ID} \) in the Selling Equilibrium in (89) and \( \theta_{DI} \) in the Buying Equilibrium in (94).

For now, we restrict attention to where  \( e < n^{D} + n^{LD} \). Later on, we will verify that the condition is necessary for the existence of the Balanced Equilibrium. For the solution of (101) to be a valid equilibrium, it has to be such that (a)  \( n_{0}^{SD} \) as given by (99) satisfies  \( n_{0}^{SD} \in [0, n^{SD}] \) for  \( e < n^{LD} \) and  \( n_{0}^{SD} \in (e - n^{LD}, n^{D} - e) \) for  \( e \in \left[ n^{LD}, \frac{n^{D} + n^{LD}}{2} \right] \) and (b)  \( \eta(\theta_{ID}) \in (0, 1) \), as given by (100).

By (99), where  \( e < n^{LD} \), for  \( n_{0}^{SD} \geq 0 \),

\[
\frac{e}{\eta(\theta_{DI})} \geq A - \frac{e}{\delta} - n^{D} \tag{RBA.1}
\]

has to hold. The condition is guaranteed to hold if

\[
A - \frac{e}{\delta} - n^{D} \leq e. \tag{ABA.1a}
\]

Otherwise, (RBA.1) can only hold if the LHS of (101) is non-negative when evaluated at  \( \theta_{DI} = \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - n^{D}} \right) \); i.e.,

\[
A - \frac{e}{\delta} \leq n^{D} + \frac{n^{LD}}{\mu^{-1} \left( \frac{e}{A - \frac{e}{\delta} - n^{D}} \right)} = B_{L}. \tag{ABA.1b}
\]
Note that (RBA.1) holds if either (ABA.1a) or (ABA.1b) is satisfied. Given that \( \frac{n^{LD}}{\mu - 1 \left( \frac{e}{\delta} \right)} < e \) by Remark 2, however, the latter condition subsumes the former one to begin with. Next, for \( n_0^{SD} \leq n^{SD} \), by (99),

\[
\frac{e}{\eta(\theta_{DI})} \leq A - \frac{e}{\delta} - n^{LD}.
\]

(RBA.2)

The condition holds if the LHS of (101) is non-positive when evaluated at \( \theta_{DI} = \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - n^{LD}} \right) \); i.e.,

\[
A - \frac{e}{\delta} \geq n^{LD} + \frac{n^D}{\mu - 1 \left( \frac{e}{n^{LD}} \right)} = B_M.
\]

(ABA.2)

Where \( e \in \left[ n^{LD}, \frac{n^{D} + n^{LD}}{2} \right) \), for \( n_0^{SD} > e - n^{LD} \),

\[
\frac{e}{\eta(\theta_{DI})} > A - \frac{e}{\delta} - n^D - n^{LD} + e
\]

(RBA.3)

has to hold. The condition is guaranteed to hold if

\[
A - \frac{e}{\delta} - n^D - n^{LD} \leq 0.
\]

Otherwise, (RBA.3) can only hold if the LHS of (101) is positive when evaluated at \( \theta_{DI} = \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - n^{LD} + e} \right) \); i.e.,

\[
\eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - n^{LD} + e} \right) - \frac{e}{A - \frac{e}{\delta} - n^{LD} + e} > 0.
\]

This inequality is met for sure by Remark 1, as \( A - \frac{e}{\delta} - n^{LD} + e > 0 \) implies \( A - \frac{e}{\delta} - n^{LD} + e < 1 \). Next, for \( n_0^{SD} < n^D - e \), by (99),

\[
\frac{e}{\eta(\theta_{DI})} < A - \frac{e}{\delta} - e,
\]

(RBA.4)

By (100), the condition for \( \eta(\theta_{ID}) \in (0,1) \) is the same condition as (RBA.4). The condition holds if the LHS of (101) is negative when evaluated at \( \theta_{DI} = \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - e} \right) \); i.e.,

\[
\frac{n^D + n^{LD} - e}{A - \frac{e}{\delta} - e} - \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - e} \right) > 0.
\]

(102)

The condition can only hold for \( e < \frac{n^D + n^{LD}}{2} \), justifying our previous claim that the Balanced Equilibrium can only hold for \( e \) bounded from below the given value, because, otherwise,

\[
\frac{n^D + n^{LD} - e}{A - \frac{e}{\delta} - e} \leq \frac{e}{A - \frac{e}{\delta} - e} < \eta^{-1} \left( \frac{e}{A - \frac{e}{\delta} - e} \right),
\]

where the last inequality is by Remark 1. Given \( e < \frac{n^D + n^{LD}}{2} \), (102) is equivalent to

\[
A - \frac{e}{\delta} > e + \frac{n^D + n^{LD} - e}{\mu^{-1} \left( \frac{e}{n^D + n^{LD} - e} \right)} = B_M.
\]

(ABA.4)
The condition for there to be more sellers than buyers among large dealers in the inter-dealer market (38), by (34) and (36), and \( n_0^{LD} = n_2^{LD} = 0 \) in the Balanced Equilibrium, simplifies to
\[
\mu(\theta_{DI}) \geq \eta(\theta_{ID}).
\]
By (100),
\[
\mu(\theta_{DI}) - \eta(\theta_{ID}) = \mu(\theta_{DI}) \frac{(A - e/\delta) \eta(\theta_{DI}) - e - e\theta_{DI}}{(A - e/\delta) \eta(\theta_{DI}) - e}.
\]
The denominator is guaranteed positive for \( \eta(\theta_{ID}) \in [0, 1] \). The expression then has the same sign as the numerator; i.e.,
\[
\mu(\theta_{DI}) - \eta(\theta_{ID}) \geq 0 \Leftrightarrow \eta(\theta_{DI}) \left( A - \frac{e}{\delta} \right) - e - e\theta_{DI} \geq 0.
\]
Rewrite (101) as
\[
\eta(\theta_{DI}) \left( A - \frac{e}{\delta} \right) - e - e\theta_{DI} = \eta(\theta_{DI}) (n^{D} + n^{LD}) - 2e\theta_{DI}.
\]
We seek conditions on how the two sides of the equation meet at a non-negative value.

Properties of \( \eta(\theta) \left( A - \frac{e}{\delta} \right) - e - e\theta \)
1. equal to \(-e\) at \( \theta = 0 \),
2. tends to negative infinity as \( \theta \to \infty \),
3. given condition \( ABA.4 \), so that \( A - \frac{e}{\delta} - e > 0 \), is inverted-U,
4. if \( \max \{ \eta(\theta) \left( A - \frac{e}{\delta} \right) - e - e\theta \} > 0 \), rises above zero for a range of \( \theta \).

Properties of \( \eta(\theta) (n^{D} + n^{LD}) - 2e\theta \)
1. equal to 0 at \( \theta = 0 \),
2. tends to negative infinity as \( \theta \to \infty \).
3. For \( e < \frac{n^{D} + n^{LD}}{2} \), is inverted-U.

Given these properties of the two sides of (104), the RHS is greater than the LHS before the two sides meet, whereas the LHS is less than the RHS thereafter. Then, if at where the RHS vanishes, i.e.,
\[
\theta = \mu^{-1} \left( \frac{2e}{n^{D} + n^{LD}} \right),
\]
the LHS is non-negative; i.e.,
\[
A - \frac{e}{\delta} \geq \frac{n^{D} + n^{LD}}{2} + \frac{\frac{n^{D} + n^{LD}}{2}}{\mu^{-1} \left( \frac{2e}{n^{D} + n^{LD}} \right)} = S,
\]
then the meeting point is where the two sides are non-negative.

If \( ABA.S \) holds, the relevant inter-dealer market equilibrium condition is (39), which becomes
\[
n^{LD} (\mu(\theta_{DI}) - \eta(\theta_{ID})) \leq (n_0^{SD} (1 - \mu(\theta_{DI})) + n_1^{SD} \eta(\theta_{ID})),
\]
51
after substituting in (32), (34), and (36). By (98)-(100), the condition becomes
\[
\left(n^D - A + \frac{e}{\delta} + \frac{e}{\eta(\theta_{DI})}\right)(1 - \mu(\theta_{DI})) + \left(\frac{e}{\mu(\theta_{DI})} - n^{LD}\right) \mu(\theta_{DI}) \geq 0. \tag{RBA.5}
\]
Rewrite (101) as
\[
\frac{e}{\mu(\theta_{DI})} - n^{LD} = n^D - A + \frac{e}{\delta} + \frac{e}{\eta(\theta_{DI})}.
\]
The LHS of (RBA.5) is a weighted average of the two terms in this equation. Thus, if the equation holds where the two sides are non-negative, (RBA.5) must hold. In turn, in case \(e < n^{LD}\) and if (RBA.1) holds, under which \(n^0_{SD} \geq 0\), and in case \(e \in \left[n^{LD}, \frac{n^D + n^{LD}}{2}\right]\) and if (RBA.3) holds, under which \(n^0_{SD} > e - n^{LD}\), the RHS of the equation is guaranteed non-negative.

If (ABA.5) holds in reverse, the relevant inter-dealer market equilibrium condition is (40), which becomes
\[
n^{LD} (\eta(\theta_{ID}) - \mu(\theta_{DI})) \leq n^0_{SD} \mu(\theta_{DI}) + n^1_{SD} (1 - \eta(\theta_{ID})),
\]
after substituting in (33), (34), and (36). By (98)-(100), the condition becomes
\[
\left(A - n^{LD} - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})}\right)(1 - \mu(\theta_{DI})) + \left(n^D - \frac{e}{\mu(\theta_{DI})}\right) \mu(\theta_{DI}) \geq 0. \tag{RBA.6}
\]
Rewrite (101) as
\[
A - n^{LD} - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})} = n^D - \frac{e}{\mu(\theta_{DI})}.
\]
The LHS of (RBA.6) is a weighted average of the two terms in this equation. Thus if the equation holds at the point where the two sides are non-negative, (RBA.6) must hold. In turn, in case \(e < n^{LD}\) and if (RBA.2) holds, under which \(n^0_{SD} \leq n^{SD}\) and in case \(e \in \left[n^{LD}, \frac{n^D + n^{LD}}{2}\right]\) and if (RBA.4) holds, under which \(n^0_{SD} < n^D - e\), the LHS of the equation is guaranteed non-negative.

Notice that in case \(e < n^{LD}\), (RBA.2) is a more stringent condition than (RBA.4). Then, for \(\theta_{ID}\) and \(\theta_{DI}\) defined by (100) and (101) to be a valid Balanced Equilibrium, it suffices that (ABA.1b) and (ABA.2) hold; i.e., \(A - \frac{\delta}{\epsilon} \in [B_M, B_L]\). Otherwise for \(e \in \left[n^{LD}, \frac{n^D + n^{LD}}{2}\right]\), the equilibrium holds under (ABA.4); i.e., \(A - \frac{\delta}{\epsilon} > B_M\). In either case, for \(A - \frac{\delta}{\epsilon} \leq S\), small dealers sell in equilibrium; otherwise small dealers buy.

**Ranking of the Bounds** That \(B_S \leq B_M\) follows from \(e \leq n^{LD}\), whereas that \(B_M \leq S \leq B_L\) follows from \(n^{LD} \leq n^D\) and Remark 3. That \(B_M \leq S\) follows from \(e \leq n^{LD} + \frac{n^{SB}}{2}\) and Remark 3.
Proof of Corollary 1  The condition $e \in \left[ n^{LD}, n^{LD} + \frac{n^{SD}}{2} \right]$ is equivalent to $n^{LD} \in (2e - n^D, e]$. For such $n^{LD}$, the Balanced Equilibrium indeed holds if the condition in Proposition 2(b) is met, which can be rewritten as the first condition of the Proposition. Notice that the RHS of the condition is greater than $2e - n^D$ by Remark 1 in the proof of Proposition 2, meaning that any $n^{LD}$ that satisfies the condition exceeds $2e - n^D$. Now when $n^{LD}$ rises up to $e$, Proposition 2(a) applies. At $n^{LD} = e$, $B_L \to \infty$ and $B_M = B_S$, in which case the Balanced Equilibrium continues to hold. At $n^{LD} = n^D$, 

$$B_M = S = B_L = B,$$

in which case the Balanced Equilibrium still holds only if $A - e/\delta = B$. Otherwise, for $A - e/\delta < (>) B$, the Selling (Buying) Equilibrium holds. In general, as $n^{LD}$ increases from $e$ to $n^D$, $B_L$ falls from infinity, whereas $B_M$ and $S$ go up and diverge from $B_S$. Eventually the three bounds converge to $B$. Then, for $A - e/\delta < (>) B$, the Balanced Equilibrium must turn into the Selling (Buying) Equilibrium at some $n^{LD} \in (e, n^D)$. The cutoff values are from Proposition 2(a).

Proof of Corollary 2  For $e < n^{LD}$, in both the Selling Equilibrium, which holds for $A - e/\delta \in (B_S, B_M)$, and in the Balanced Equilibrium for $A - e/\delta \in [B_M, S)$, small dealers sell to large dealers. In both the Balanced Equilibrium for $A - e/\delta \in (S, B_L)$ and the Buying Equilibrium, which holds for $A - e/\delta \geq B_L$, small dealers buy from large dealers. For $e \in \left[ n^{LD}, n^{LD} + \frac{n^{SD}}{2} \right)$, only the Balanced Equilibrium can hold, in which the cutoff value for $A - e/\delta$ is similarly $S$.

Proof of Lemma 3  In this equilibrium, all large dealers enter the investor-dealer market as an $L_1$. Write $\pi (L_1, I)$ as the probability that an $L_1$ will exit the investor-dealer market as an $I = I_0, L_1, L_2,$ and $L_3$ and let $\eta = \eta (\theta_{ID})$ and $\mu = \mu (\theta_{ID})$ be the respective matching probabilities of a dealer-seller and a dealer-buyer. Then,

$$\pi (L_1, L_1) = 1 - \pi (L_1, L_0) - \pi (L_1, L_2) - \pi (L_1, L_3) = 1 - \alpha \eta (1 - \eta) - (\alpha \mu (1 - \alpha) + (1 - \alpha \eta) \alpha \mu) - \mu (1 - \alpha) (1 - \alpha \eta) = 1 + \alpha^2 \mu \eta + \alpha \mu \eta - \alpha \eta - \mu.$$

To understand the second line, take $\pi (L_1, L_0)$ for example. An $L_1$ has one unit inventory which allows him to only sell to a small investor and two units spare capacity which allows him to buy from either a small or a large investor. In this case, he will become an $L_0$ if he sells his unit inventory to a small investor and does not buy from any investor at all. The fraction of large dealers not trading in the inter-dealer market is simply $\pi (L_1, L_1)$.

Let $\psi$ be the fraction of small dealers entering the investor-dealer market as an $S_1$ and $1 - \psi$ be the fraction entering the market as an $S_0$. The probabilities that a given small dealer will exit the investor-dealer market as an $S_0$ and an $S_1$, respectively, are

$$\pi (S_0) = (1 - \psi) \pi (S_0, S_0) + \psi \pi (S_1, S_0) = (1 - \psi) (1 - \mu) + \psi (1 - \alpha \mu) \alpha \eta.$$
\[ \pi(S_1) = (1 - \psi) \pi(S_0, S_1) + \psi \pi(S_1, S_1) \]
\[ = (1 - \psi) \pi(S_0, S_1) + \psi (1 - \pi(S_1, S_0) - \pi(S_1, S_2)) \]
\[ = (1 - \psi) \alpha \mu + \psi (1 - \alpha \eta (1 - \alpha \mu) - \alpha \mu (1 - \alpha \eta)) \]
\[ = (1 - \psi) \alpha \mu + \psi (1 - \alpha (\eta + \mu - 2\alpha \mu \eta)). \]

Let \( \phi \) be the fraction of \( S_0 \)s buying in the inter-dealer market. Then, the fraction of small dealers not trading in the market is \( (1 - \phi) \pi(S_0) + \pi(S_1) \). We seek to show that
\[ (1 - \phi) \pi(S_0) + \pi(S_1) > \pi(L_1, L_1). \quad (106) \]

To proceed, we next calculate \( \psi \). To this end, note that

1. The fractions of small dealers entering inter-dealer market as an \( S_1 \) and therefore will exit as an \( S_1 \) is equal to
\[ \psi \pi(S_1, S_1) + (1 - \psi) \pi(S_0, S_1) \]
\[ = \psi (1 - \alpha (\eta + \mu - 2\alpha \mu \eta)) + (1 - \psi) \mu \alpha. \]

2. All \( S_2 \)s entering the market will also exit as an \( S_1 \). That fraction is
\[ (1 - \psi) \pi(S_0, S_2) + \psi \pi(S_1, S_2) \]
\[ = (1 - \psi) \mu (1 - \alpha) + \psi (1 - \alpha \mu) \alpha \mu \]

3. A fraction \( \phi \) of \( S_0 \)s entering the market will exit as \( S_1 \)s. That fraction is
\[ \phi ((1 - \psi) \pi(S_0, S_0) + \psi \pi(S_1, S_0)) \]
\[ = \phi ((1 - \psi) (1 - \mu) + \psi (1 - \alpha \mu) \alpha \eta) \]

The sum of (1), (2), and (3) is equal to \( \psi \). Solving the equation for
\[ \psi = \frac{\mu + (1 - \mu) \phi}{\mu + (1 - \mu) \phi + (1 - \phi) (1 - \alpha \mu) \alpha \eta}. \quad (107) \]

Expanding the LHS of (106) and then by virtue of (107),
\[ (1 - \psi) \left( \alpha \mu + (1 - \phi) \left( (1 - \mu) + \frac{\psi}{1 - \psi} (1 - \alpha \mu) \alpha \eta \right) \right) \]
\[ + \psi (1 - \alpha (\eta + \mu - 2\alpha \mu \eta)) \]
\[ = (1 - \psi) (\alpha \mu + 1) + \psi (1 - \alpha (\eta + \mu - 2\alpha \mu \eta)). \]

Obviously, \( \alpha \mu + 1 > \pi(L_1, L_1) \) but it can also be shown that \( 1 - \alpha (\eta + \mu - 2\alpha \mu \eta) > \pi(L_1, L_1) \) as well. Then, (106) is guaranteed to hold.
Proof of Lemma 4  First, that all dealers are dealer-buyers and dealer-sellers and that $\alpha$ measures of small investors buy and sell in each period imply that

$$\alpha e = n^D \eta(\theta_{ID}) \alpha,$$

$$\alpha e = n^D \mu(\theta_{DI}) \alpha.$$ 

The two conditions combine to yield $\eta(\theta_{ID}) = \mu(\theta_{DI})$, which in turn implies that $m_0^{SD} = m_2^{SD}$, given that

$$m_0^{SD} = \eta(\theta_{ID}) \alpha (1 - \mu(\theta_{DI}) \alpha n^{SD},$$

$$m_2^{SD} = \mu(\theta_{DI}) \alpha (1 - \eta(\theta_{ID}) \alpha n^{SD}.$$ 

Next, with $(1 - \alpha) e$ measure of large investors buying from large dealers holding a two-unit inventory as well as $(1 - \alpha) e$ measure of large investors selling to large dealers holding a one-unit inventory,

$$(1 - \alpha) e = n_2^{LD} \eta(\theta_{ID}) (1 - \alpha),$$

$$(1 - \alpha) e = n_1^{LD} \mu(\theta_{DI}) (1 - \alpha).$$

Given that $\eta(\theta_{ID}) = \mu(\theta_{DI})$, the above imply $n_1^{LD} = n_2^{LD} = n^{LD}/2$. To follow then is $m_0^{LD} = m_3^{LD}$ given that

$$m_0^{LD} = n_1^{LD} \eta(\theta_{ID}) \alpha (1 - \mu(\theta_{DI})) + n_2^{LD} \eta(\theta_{ID}) (1 - \alpha) (1 - \alpha \mu(\theta_{DI})),$$

$$m_3^{LD} = n_1^{LD} \mu(\theta_{DI}) (1 - \alpha) (1 - \alpha \eta(\theta_{ID})) + n_2^{LD} \mu(\theta_{DI}) \alpha (1 - \eta(\theta_{ID})).$$

Proof of Lemma 5  The surpluses of the possible trades are as follows.

$$z_{I_B,S_1} = U_H^{ON} - U^B - (V_1^{SD} - V_0^{SD}),$$

$$z_{I_B,L_i} = U_H^{ON} - U^B - (V_i^{LD} - V_{i-1}^{LD}) \text{ for } i = 1, 2,$$

$$z_{S_0,I_S} = V_1^{SD} - V_0^{SD} - U_L^{ON},$$

$$z_{L_i,I_S} = V_{i+1}^{LD} - V_i^{LD} - U_L^{ON} \text{ for } i = 0, 1,$$

$$z_{S_0,L_i} = V_1^{SD} - V_0^{SD} - (V_i^{LD} - V_{i-1}^{LD}) \text{ for } i = 1, 2,$$

$$z_{L_0,S_1} = V_{i+1}^{LD} - V_i^{LD} - (V_1^{SD} - V_0^{SD}) \text{ for } i = 0, 1,$$

$$z_{L_0,L_2} = V_1^{LD} - V_0^{LD} - (V_2^{LD} - V_1^{LD}),$$

$$z_{L_1,L_1} = V_2^{LD} - V_1^{LD} - (V_1^{LD} - V_0^{LD}).$$

An $S_1$ can sell to an $IB$ (investor-buyer), an $L_0$, or an $L_1$. He will not sell to an $IB$ only if selling to other dealers yields a strictly larger surplus; i.e.,

$$\max \{z_{L_0,S_1}, z_{L_1,S_1}\} > z_{I_B,S_1}.$$ 

Expanding the expressions for the $zs$,

$$\max \{V_1^{LD} - V_0^{LD}, V_2^{LD} - V_1^{LD}\} > U_H^{ON} - U^B.$$ 

55
Subtracting $U_{ON} - U^{IB}$ from the two sides of the condition
\[
\max \{ V_1^{LD} - V_0^{LD} - (U_{ON} - U^{B}) , V_2^{LD} - V_1^{LD} - (U_{ON} - U^{B}) \} > 0.
\]
The two terms inside the max operator are simply the negatives of $z_{IB,L_1}$ and $z_{IB,L_2}$, respectively. Then, the condition becomes
\[
\max \{ -z_{IB,L_1}, -z_{IB,L_2} \} > 0 \Leftrightarrow \min \{ z_{IB,L_1}, z_{IB,L_2} \} < 0.
\]
All this implies that if one type of dealer-seller finds it optimal not to sell to investor-buyers, then only one type of dealer-seller may find it optimal to do so. In any active steady-state equilibrium, indeed at least one type of dealer-seller must do so.

Now, suppose only $S_1$s sell to $IB$ where
\[
z_{IB,S_1} = U_{ON} - U^{B} - (V_1^{SD} - V_0^{SD}) \geq 0. \tag{108}
\]
An $L_1$ may then only sell to an $S_0$ or another $L_1$. Selling to an $S_0$ is optimal if
\[
z_{S_0,L_1} = V_1^{SD} - V_0^{SD} - (V_1^{LD} - V_0^{LD}) \geq 0.
\]
But if the condition holds,
\[
z_{IB,L_1} = U_{ON} - U^{B} - (V_1^{LD} - V_0^{LD}) \geq 0
\]
must hold given (108). The hypothesis that only $S_1$ sell to $IB$ then implies that selling to another $L_1$ must be optimal for the $L_1$ (otherwise the $L_1$ has no one to sell to), where
\[
z_{L_1,L_1} = V_2^{LD} - V_1^{LD} - (V_1^{LD} - V_0^{LD}) \geq 0. \tag{109}
\]
An $L_2$ may sell to an $S_0$ or an $L_0$ if selling to an $IB$ is not optimal. Selling to an $S_0$ is optimal if
\[
z_{S_0,L_2} = V_1^{SD} - V_0^{SD} - (V_2^{LD} - V_1^{LD}) \geq 0.
\]
But if the condition holds,
\[
z_{IB,L_2} = U_{ON} - U^{B} - (V_2^{LD} - V_1^{LD}) \geq 0
\]
must hold given (108). The hypothesis that only $S_1$ sell to $IB$ then implies that selling to an $L_0$ must be optimal for the $L_2$, where
\[
z_{L_0,L_2} = V_1^{LD} - V_0^{LD} - (V_2^{LD} - V_1^{LD}) \geq 0. \tag{110}
\]
The two conditions, (109) and (110), together imply that
\[
V_1^{LD} - V_0^{LD} = V_2^{LD} - V_1^{LD}.
\]
Thus, if neither $L_1$s nor $L_2$s find it optimal to sell to investor-buyers or to small dealers, large dealers do not gain by selling and buying among themselves either. They must then be inactive in equilibrium.

56
Next, suppose only $L_1$s sell to investor-buyers, where
\[ z_{IB,L_1} = U_{R1}^{ON} - U^B - (V_1^{LD} - V_0^{LD}) \geq 0. \]  
(111)

An $S_1$ may sell to an $L_0$ or to an $L_1$ if not selling to an investor-buyer. If the first sale is optimal, it must be optimal for the $S_1$ to sell to an $IB$ as well given (111). The hypothesis that only $L_1$ sells to investor-buyers then requires that it is optimal for an $S_1$ to sell to an $L_1$ where
\[ z_{L_1,S_1} = V_2^{LD} - V_1^{LD} - (V_1^{SD} - V_0^{SD}) \geq 0. \]  
(112)

An $L_2$ may sell to an $L_0$ or to an $S_0$. If the first sale is optimal, it must be optimal for the $L_2$ to sell to an $IB$ as well given (111). The condition for the second sale to be optimal is that
\[ z_{S_0,L_2} = V_1^{SD} - V_0^{SD} - (V_2^{LD} - V_1^{LD}) \geq 0. \]  
(113)

The two conditions, (112) and (113), together imply that
\[ V_1^{SD} - V_0^{SD} = V_2^{LD} - V_1^{LD}. \]

Thus, if neither $S_1$s nor $L_2$s find it optimal to sell to investor-buyers, $S_1$s only sell to $L_1$s, where such trades do not yield any surplus. This implies that small dealers must be inactive in equilibrium.

The case for where only $L_2$s sell to investor-buyers can be shown in a similar way to imply that small dealers must be inactive in equilibrium.

The proof that in any equilibrium in which both small and large dealers are active, investor-sellers must sell to all three types of dealer-buyers can be constructed similarly.

**Proof of Proposition 3** With Nash Bargaining and each agent in a match entitled to one-half of the match’s surplus, we can rewrite dealers’ value functions as follows.

- \[ rV_0^{SD} = \mu(\theta_{DI}) \frac{z_{S_0,L_2}}{2} + \alpha \left\{ \frac{n_1^{LD}}{2n_D} \max\{z_{S_0,L_1},0\} + \frac{n_2^{LD}}{2n_D} \max\{z_{S_0,L_2},0\} \right\}, \]
- \[ rV_1^{SD} = \eta(\theta_{ID}) \frac{z_{IB,S_1}}{2} + \alpha \left\{ \frac{n_0^{LD}}{2n_D} \max\{z_{L_0,S_1},0\} + \frac{n_1^{LD}}{2n_D} \max\{z_{L_1,S_1},0\} \right\}, \]
- \[ rV_0^{LD} = \mu(\theta_{DI}) \frac{z_{L_0,L_2}}{2} + \alpha \left\{ \frac{n_1^{SD}}{2n_D} \max\{z_{L_0,S_1},0\} + \frac{n_2^{LD}}{2n_D} \max\{z_{L_0,L_2},0\} \right\}, \]
- \[ rV_1^{LD} = \mu(\theta_{DI}) \frac{z_{L_1,L_2}}{2} + \eta(\theta_{ID}) \frac{z_{IB,L_1}}{2} + \alpha \left\{ \frac{n_0^{SD}}{2n_D} \max\{z_{S_0,L_1},0\} + \frac{n_1^{SD}}{2n_D} \max\{z_{L_1,S_1},0\} + \frac{n_2^{LD}}{2n_D} \max\{z_{L_1,L_1},0\} \right\}, \]
- \[ rV_2^{LD} = \eta(\theta_{ID}) \frac{z_{IB,L_2}}{2} + \alpha \left\{ \frac{n_0^{SD}}{2n_D} \max\{z_{S_0,L_2},0\} + \frac{n_0^{LD}}{2n_D} \max\{z_{L_0,L_2},0\} \right\}. \]
Suppose \( V_1^{SD} - V_0^{SD} > V_1^{LD} - V_0^{LD} \). Then \( z_{I_B,S_1} < z_{I_B,L_1} \) and \( z_{S_0,I_S} > z_{L_0,I_S} \). Together with the fact that \( z_{L_1,I_S} \geq 0 \), this implies that
\[
\frac{\eta(ID)}{2} z_{I_B,S_1} - \mu(ID) \frac{z_{S_0,I_S}}{2} < \mu(ID) \frac{z_{L_1,I_S}}{2} + \frac{\eta(ID)}{2} z_{I_B,L_1} - \mu(DI) \frac{z_{L_0,I_S}}{2}.
\]
Also, \( V_1^{SD} - V_0^{SD} > V_1^{LD} - V_0^{LD} \) implies that \( z_{S_0,L_1} > 0 > z_{L_0,S_1}, z_{S_0,L_2} > z_{L_0,L_2}, \) and \( z_{L_1,S_1} < z_{L_1,L_1} \). This means
\[
\frac{n_1^{LD}}{2n^D} \max\{z_{S_0,L_1}, 0\} + \frac{n_2^{LD}}{2n^D} \max\{z_{S_0,L_2}, 0\} > \frac{n_1^{SD}}{2n^D} \max\{z_{L_0,S_1}, 0\} + \frac{n_2^{SD}}{2n^D} \max\{z_{L_0,L_2}, 0\}
\]
and
\[
\frac{n_0^{LD}}{2n^D} \max\{z_{L_0,S_1}, 0\} + \frac{n_1^{LD}}{2n^D} \max\{z_{L_1,S_1}, 0\} < \frac{n_0^{SD}}{2n^D} \max\{z_{S_0,L_1}, 0\} + \frac{n_1^{SD}}{2n^D} \max\{z_{S_0,L_2}, 0\} + \frac{n_1^{LD}}{2n^D} \max\{z_{L_1,L_1}, 0\}
\]
The above three inequalities together imply that \( V_1^{SD} - V_0^{SD} < V_1^{LD} - V_0^{LD} \). This is a contradiction.

Now suppose \( V_2^{LD} - V_1^{LD} > V_1^{SD} - V_0^{SD} \). Similarly, we can show that this implies \( z_{L_1,I_S} > z_{S_0,I_S}, z_{I_B,L_2} < z_{I_B,S_1}, z_{S_0,L_2} < 0 < z_{L_0,S_1} \) and \( z_{L_0,L_2} < z_{L_0,S_1} \). These inequalities in turn imply that \( V_2^{LD} - V_1^{LD} < V_1^{SD} - V_0^{SD} \). This is a contradiction.

Given that we have shown \( V_1^{LD} - V_0^{LD} \geq V_1^{SD} - V_0^{SD} \geq V_2^{LD} - V_1^{LD} \), it is straightforward to verify that the two equalities hold are strict unless \( z_{I_B,L_1} = 0 \).

**Proof of Proposition 4a** In the Selling Equilibrium, \( \theta_{DI} \) is implicitly given by (87), in which \( A \) is absent. By (89), \( \theta_{ID} \) is decreasing in \( A \) given that \( \theta_{DI} \) does not vary with \( A \).

In the Balanced Equilibrium, \( \theta_{DI} \) is implicitly given by (101), the solution to which is at a point where the LHS of the equation is increasing. In the meantime, the LHS of the equation is increasing in \( A \). Then, \( \partial \theta_{DI} / \partial A < 0 \). To evaluate the effect of \( A \) on \( \theta_{ID} \), first rewrite (101) as
\[
A - \frac{e}{\delta} = n^D + n^{LD} - \frac{e (\theta_{DI} - 1)}{\eta(\theta_{DI})}.
\]
Then substitute the equation into (100) to yield
\[
\eta(\theta_{ID}) = \frac{\eta(\theta_{DI})}{(n^D + n^{LD}) \eta(\theta_{DI})} - \frac{e}{\delta} \theta_{DI},
\]
the RHS of which is increasing in \( \theta_{DI} \) due to the concavity of \( \eta \). Then, \( \theta_{ID} \) must be decreasing in \( A \).

In the Buying Equilibrium, \( \theta_{ID} \) is implicitly given by (91), in which \( A \) is absent, whereas \( \theta_{DI} \) is given by the same equation that defines \( \theta_{DI} \) in the Balanced Equilibrium.

The continuity can be established by verifying that the equations for \( \theta_{DI} \) and \( \theta_{ID} \) for one equilibrium type coincide with another at each of the two cutoff values of \( A - e/\delta \).
Proof of Proposition 4b  By (33), (36), the restrictions in the first column of Table 2, and (87), in the Selling Equilibrium,

\[ TV = \frac{e}{n^D} \left( n^D - n^{LD} + e \left( \frac{1}{\eta�}\right) - 1 \right). \]  

(120)

The result of the Proposition then follows, given that, by Proposition 4a, \( \theta_{ID} \) is decreasing in \( A \) in the Selling Equilibrium. By (36) and (34) and the restrictions in the second column of Table 2, in the Balanced Equilibrium,

\[ TV = \begin{cases} 
    n^{LD} \eta (\theta_{ID})(1 - \mu (\theta_{DI})) & A \leq S + \frac{e}{\delta} \\
    n^{LD} \mu (\theta_{DI})(1 - \eta (\theta_{ID})) & A > S + \frac{e}{\delta}. 
\end{cases} \]  

(121)

The result of the Proposition then follows given that, by Proposition 4a, both \( \theta_{ID} \) and \( \theta_{ID} \) are decreasing in \( A \) in the Selling Equilibrium. By (32), (34), the restrictions in the third column of Table 2, and (91), in the Buying Equilibrium,

\[ TV = \frac{e}{n^D} \left( n^D - n^{LD} + e \left( \frac{1}{\mu (\theta_{DI})} - 1 \right) \right). \]  

(122)

The result of the Proposition then follows given that, by Proposition 4a, \( \theta_{DI} \) is decreasing in \( A \) in the Buying Equilibrium.

Evaluate (120) and the first line of (121) at where \( A = B_M + e/\delta \) and (91) yields the same value of \( e \left( 1 - \frac{e}{n^D} \right) \). Evaluate the second line of (121) and (122) at where \( A = B_L + e/\delta \) and (92) yields the same value of \( e \left( 1 - \frac{e}{n^D} \right) \). This proves continuity.

Proof of Proposition 4c  In the Selling Equilibrium, \( p \) is given by (78), which is increasing in \( \theta_{ID} \) and \( \theta_{DI} \). Given that in the Selling Equilibrium, \( \theta_{ID} \) is decreasing in \( A \) but \( \theta_{DI} \) is independent of \( A \), \( p \) must be decreasing in \( A \). That there is a discrete fall in \( p \) as the Selling Equilibrium turns into the Balanced Equilibrium can be established by showing that the denominator of (80), which gives \( p \) in the Balanced Equilibrium, is larger than that of (78) at any \( \theta_{ID} \) and \( \theta_{DI} \). Moreover, by (80), \( p \) in the Balanced Equilibrium is also increasing in \( \theta_{ID} \) and \( \theta_{DI} \), both of which are decreasing in \( A \). Finally, that there is a discrete fall in \( p \) as the Balanced Equilibrium turns into the Buying Equilibrium can be established by noting that the numerator of (80) always stays strictly positive.

Proof of Proposition 5a  In the Balanced Equilibrium, \( \theta_{DI} \) is given by the solution to (101), whereas \( \theta_{ID} \) can be recovered from (100) once \( \theta_{DI} \) is known from the former equation. In the Buying Equilibrium, \( \theta_{DI} \) and \( \theta_{ID} \) are given by the solutions to (94) and (91), respectively. In the Selling Equilibrium, \( \theta_{DI} \) is given by the solution to (87), whereas \( \theta_{ID} \) can be recovered from (89) once \( \theta_{DI} \) is known from the former equation. The comparative steady states followed straightforwardly from these equations. Just as in the proof of Proposition 3a, the continuity can be established by verifying that the equations for \( \theta_{DI} \) and \( \theta_{ID} \) for one equilibrium type coincide with another at each of the two cutoff values of \( A - e/\delta \).
Proof of Proposition 5b In the Selling Equilibrium, $TV$, given by (120) is decreasing in $n^{LD}$, given that $\theta_{ID}$ is independent of $n^{LD}$ in the Selling Equilibrium. In the Buying equilibrium, $TV$, given by (122) can be shown to be decreasing in $n^{LD}$ with $\theta_{DI}$ given in (94).

In the Balanced Equilibrium, $TV$ is given by either the first or the second line of (121). To show that both expressions are increasing in $n^{LD}$, we begin with noting that $A^D$, by (83) and (84), in the first instance is given by

$$A^D = A - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})},$$

But by (101),

$$A - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})} = n^{LD} + n^D - \frac{e}{\mu(\theta_{DI})}$$

Because $\theta_{DI}$ increases with $n^{LD}$ in the Balanced Equilibrium, the LHS strictly increases with $n^{LD}$, and so dealers hold more inventory in total. The RHS, however, can only rise by less than the increase in $n^{LD}$. Given that in the Balanced Equilibrium, $A^D = n^{SD} + n^{LD}$, a larger $n^{LD}$ must be accompanied by a smaller $n^{SD}$. Also, according to (99), there would also have to be a smaller $n^{SD}$. In the investor-dealer market, both dealer-sellers and dealer-buyers execute $e$ trades in the steady state; i.e.,

$$(n^{LD} + n^{SD})\eta(\theta_{ID}) = (n^{LD} + n^{SD})\mu(\theta_{DI}) = e$$

Because $n^{SD}\eta(\theta_{ID})$ and $n^{SD}\mu(\theta_{DI})$ strictly decrease with $n^{LD}$, $n^{LD}\eta(\theta_{ID})$ and $n^{LD}\mu(\theta_{DI})$ must be strictly increasing in $n^{LD}$. This implies that both the first and the second lines of (121) are strictly increasing in $n^{LD}$.

The proof of continuity is as in Proposition 3c.

Proof of Proposition 5c In the Selling Equilibrium, $p$ is given by (78), which does not directly depend on $n^{LD}$, given $\theta_{ID}$ and $\theta_{DI}$. But then, the two market tightness in the Selling Equilibrium do not vary with $n^{LD}$. The proof for the jump in $p$ that occurs when the Balanced Equilibrium gives way to the Buying or the Selling Equilibrium follow from Proposition 4b.

Proof of Lemma A1 The equation for $p_{I_s}$ is from combining (21) and (72). The equations for $p_{I_b}$ are from combining (23) and (77) for the Selling Equilibrium, (23), (72), and (79) for the Balanced Equilibrium, and (23), (82), and $p = 0$ for the Buying Equilibrium.

Proof of Propositions A1 and A2 By (57), given $p$, $p_{I_s}$ depends only on and is increasing in $\theta_{DI}$. In the Selling and Balanced Equilibria, $\partial \theta_{DI}/\partial A = 0$ and $\partial \theta_{DI}/\partial A < 0$, respectively. Then, $\partial p_{I_s}/\partial A$ has the same negative sign as $\partial p/\partial A$ in the two types of equilibrium. Next, in the Selling Equilibrium, $\partial \theta_{DI}/\partial n^{LD} = 0$, from which it follows that $\partial p_{I_s}/\partial n^{LD}$ has the same zero value as $\partial p/\partial n^{LD}$. In the Buying Equilibrium, by Lemma A1, $p_{I_s} = p = 0$. The discrete changes in $p_{I_s}$ at where one equilibrium type changes to another follows from the discrete changes in $p$. 
By combining (58) and (78), in the Selling Equilibrium,

\[ p_{IB} = \frac{1}{2} \beta (1 - \beta + \beta \eta (\theta_{ID})) \left( 1 - \beta + \frac{\eta(\theta_{ID})}{2} \right) v \left( 1 - \beta + \frac{\eta(\theta_{ID})}{2} \beta \right) (1 - \beta + \beta \delta - \delta \beta \eta(\theta_{ID}) \eta(\theta_{ID}) \right), \]

where \( \partial p_{IB} / \partial \theta_{ID} > 0 \). Then, given \( \partial \theta_{ID} / \partial A < 0 \), it follows that \( \partial p_{IB} / \partial A < 0 \). Meanwhile, \( \partial p_{IB} / \partial n_{LD} = 0 \) holds given \( \partial \theta_{ID} / \partial n_{LD} = \partial \theta_{DI} / \partial n_{LD} = 0 \) in the Selling Equilibrium. That \( p_{IB} \) in the Buying Equilibrium, given by (60), does not vary with \( n_{LD} \) follows from \( \partial \theta_{ID} / \partial n_{LD} = 0 \) in said equilibrium. The discrete changes in \( p_{IB} \) at which one equilibrium type changes to another can be verified by checking how, given \( \theta_{ID} \) and \( \theta_{DI} \), \( p_{IB} \) in (58) exceeds \( p_{IB} \) in (59), which in turn exceeds \( p_{IB} \) in (60).

**Proof of Lemma A2** The state variables of the planning problem (61) are \( \{ n_{H}^{ON}(t), n_{L}^{ON}(t), n_{B}^{L}(t) \} \), the initial conditions are \( \{ n_{H}^{ON}(0), n_{L}^{ON}(0), n_{B}(0) \} = \{ n_{H}^{ON}, n_{L}^{ON}, n_{B} \} \), the controls are \( \{ n_{0}^{SD}(t), n_{1}^{SD}(t), n_{0}^{LD}(t), n_{1}^{LD}(t), n_{2}^{LD}(t) \} \) and the equations of motion are, respectively,

\[ n_{H}^{ON}(t + 1) - n_{H}^{ON}(t) = -\delta n_{H}^{ON}(t) + n_{B}(t) \mu (\theta_{ID}[t]), \]
\[ n_{L}^{ON}(t + 1) - n_{L}^{ON}(t) = \delta n_{H}^{ON}(t) - n_{L}^{ON}(t) \eta (\theta_{DI}[t]), \]
\[ n_{B}(t + 1) - n_{B}(t) = \epsilon - n_{B}(t) \mu (\theta_{ID}[t]). \]

The constraints are given in (24)-(28) that hold at each moment in time, which can be summarized by the following two equations:

\[ \theta_{ID}(t) = \frac{n_{B}(t)}{n_{0}^{SD}(t) - n_{0}^{LD}(t)}, \]
\[ \theta_{DI}(t) = \frac{n_{D} + n_{LD} - A - n_{0}^{LD}(t) + n_{L}^{ON}(t) + n_{H}^{ON}(t)}{n_{L}^{ON}(t)}. \]

In the above, a pair of \( \{ n_{0}^{SD}(t), n_{0}^{LD}(t) \} \) uniquely determines the pair \( \{ \theta_{ID}(t), \theta_{DI}(t) \} \). This means that the controls can be stated in terms of the two market tightness only, whereby the admissible values are given by

\[ \theta_{ID}(t) \in \left[ \frac{n_{B}(t)}{\pi_{B}^{D}(n_{H}^{ON}(t), n_{L}^{ON}(t)), n_{L}^{ON}(t))}, \frac{n_{B}(t)}{\pi_{B}^{D}(n_{H}^{ON}(t), n_{L}^{ON}(t))}, n_{0}^{LD}(t), n_{L}^{ON}(t) \} \right], \]
\[ \theta_{DI}(t) \in \left[ \frac{\pi_{B}^{D}(n_{H}^{ON}(t), n_{L}^{ON}(t)), n_{L}^{ON}(t))}{n_{L}^{ON}(t)}, \frac{\pi_{B}^{D}(n_{H}^{ON}(t), n_{L}^{ON}(t))}{n_{L}^{ON}(t)} \right], \]

with \( \pi_{B}^{D} \) and \( n_{B}^{D} \) denoting, respectively, the largest and smallest possible measures of dealersellers and \( \pi_{B}^{D} \) and \( n_{B}^{D} \) denoting, respectively, the largest and smallest possible measures of dealer-buyers, given state variables \( n_{H}^{ON}(t) \) and \( n_{L}^{ON}(t) \). Note that:

1. To attain \( \pi_{S}^{D} \), first allocate one unit each of the assets to be held by dealers \( (A - n_{H}^{ON}(t) - n_{L}^{ON}(t)) \) to either small or large dealers, and then allocate one more unit each to large dealers if \( A - n_{H}^{ON}(t) - n_{L}^{ON}(t) > n_{D} \).

61
(2) To attain \( \pi^D_B \), first allocate two units each of the assets to be held by dealers to large dealers, and then allocate one unit each to small dealers if \( A - n_{H}^{ON}(t) - n_{L}^{ON}(t) > 2n_{LD} \). 
(3) To attain \( \pi^D_S \), first allocate one unit each of the assets to be held by dealers to large dealers, and then allocate one unit each to either large or small dealers if \( A - n_{H}^{ON}(t) - n_{L}^{ON}(t) > n_{LD} \). 
(4) To attain \( \pi^S_B \), first allocate one unit each of the assets to be held by dealers to small dealers, and then allocate two units each to large dealers if \( A - n_{H}^{ON}(t) - n_{L}^{ON}(t) > n_{SD} \).

To proceed, write (61) as

\[
W(n_{H}^{ON}(t), n_{L}^{ON}(t), n_{B}^{I}(t)) = \max_{\theta_{ID}(t), \theta_{DI}(t)} \left\{ \begin{array}{l}
n_{H}^{ON}(t) v_H \\
+ \beta W(n_{H}^{ON}(t+1), n_{L}^{ON}(t+1), n_{B}^{I}(t+1)) \end{array} \right. ,
\]

in which the state variables for \( t + 1 \) can be recovered from the equations of motions. There are four constraints corresponding to the four bounds of market tightness. Let \( \lambda_1(t) \), \( \lambda_2(t) \), \( \lambda_3(t) \) and \( \lambda_4(t) \) be the respective Lagrange multipliers of the lower and upper bounds of \( \theta_{ID}(t) \) and the lower and upper bounds of \( \theta_{DI}(t) \).

Restricting attention to the steady state, we omit all time indices in the following. The first order conditions for \( \theta_{ID}(t) \) and \( \theta_{DI}(t) \) are then given by, respectively,

\[
\beta n_B^\mu W((\theta_{ID}) (W_1 - W_3) + \lambda_1 - \lambda_2 = 0, \quad (132)
\]

\[
- \beta n_L^{ON} \eta ((\theta_{DI}) W_2 + \lambda_3 - \lambda_4 = 0. \quad (133)
\]

In addition, there are three envelope conditions, one for each state variable:

\[
W_1 = \nu_H + \beta(1 - \delta) W_1 + \beta \delta W_2 - \lambda_1 \frac{\partial(n_B^{I}/\pi^D_S)}{\partial n_H^{ON}} + \lambda_2 \frac{\partial(n_B^{I}/\pi^D_S)}{\partial n_L^{ON}}
- \lambda_3 \frac{\partial(M_B/n_L^{ON})}{\partial n_H^{ON}} + \lambda_4 \frac{\partial(M_B/n_L^{ON})}{\partial n_L^{ON}} \quad (134)
\]

\[
W_2 = \beta(1 - \eta((\theta_{DI})) W_2 - \lambda_1 \frac{\partial(n_B^{I}/\pi^D_S)}{\partial n_L^{ON}} + \lambda_2 \frac{\partial(n_B^{I}/\pi^D_S)}{\partial n_L^{ON}}
- \lambda_3 \frac{\partial(M_B/n_L^{ON})}{\partial n_L^{ON}} + \lambda_4 \frac{\partial(M_B/n_L^{ON})}{\partial n_L^{ON}} \quad (135)
\]

\[
W_3 = \beta \mu(\theta_{ID})(W_1 - W_3) - \frac{\lambda_1}{\pi^D_S} + \frac{\lambda_2}{\pi^D_S} \quad (136)
\]

We first show that \( \lambda_3 \) must equal to 0. Suppose otherwise. By the definition of \( \pi^D_S \), \( \pi^D_B \), \( \pi^D_B \), \( \lambda_1 \), \( \lambda_2 \) and \( \lambda_4 \) must all equal to 0. Then equations (132) to (136) reduce to three equations, (133), (134) and (135) with only two unknowns, \( \lambda_3 \) and \( W_2 \). The set of \( \lambda_3 \) and \( W_2 \) that satisfy all three equations is of measure zero. Therefore, \( \lambda_3 \) must equal to 0.

By the same argument, we can show that \( \lambda_2 \) must equal to 0.

Next, we prove that \( \lambda_1 \) > 0. Suppose otherwise. Together with the fact that \( \lambda_2 = 0 \), this implies \( W_1 = W_3 = 0 \). We already know \( \lambda_3 = 0 \). Then equations (132) to (136) reduce to three equations, (133), (134) and (135) with only two unknowns, \( \lambda_4 \) and \( W_2 \). We have reached the desired contradiction.
Finally, we prove that $\lambda_4 > 0$. Suppose otherwise. Then by equation (133), $W_2 = 0$. Plug it into equation (135), we must have $\lambda_1 = 0$, which contradicts our previous conclusion that $\lambda_1 > 0$.

To summarize, we have shown that $\theta_{ID} = \frac{n_H^I}{n_H^I(n_H^H,n_H^L)}$ and $\theta_{DI} = \frac{g^L(n_H^Q,n_H^Q)}{n_H^Q}$. In other words, for efficiency, we should allocate the assets held by dealers to maximize the measure of dealers holding inventory and the measure of dealers having spare capacity: first allocate one unit each to large dealers; if $A - n_H^Q - n_H^Q > n^L D$, then allocate one unit each to small dealers; if $A - n_L^Q - n_H^Q > n^D$, then allocate one more unit each to large dealers.

Proof of Proposition A3 The allocations as described in Lemma A2 are the same as the allocations as described in the discussions following Proposition 2.

References


