

Unobservable Investments and Search Frictions

Yujing Xu*

April 2016

Abstract

This paper studies investment incentives and trading efficiencies jointly in a random search framework with pre-entry investments. A seller entrant can make unobservable investments to cut production cost before searching for buyers. In the unique steady state equilibrium, investment and price dispersion emerge simultaneously with ex ante identical buyers and sellers. Despite the fact that the amount of investments is positive in equilibrium, when buyers have all the bargaining power, the equilibrium payoffs and social welfare are constant given any search friction and equal to the equilibrium values when investments are observable (indicating no investment). This property remains true even when the investment strategy becomes socially efficient as search frictions diminish to zero.

Keywords: Search Friction, Unobservable Investments, Convergence

*School of Economics and Finance, Faculty of Business and Economics, University of Hong Kong. E-mail: yujingxu@hku.hk.

The author is indebted to Hugo Hopenhayn, Simon Board, Moritz Meyer-ter-Vehn and Ichiro Obara for their guidance. The author is also grateful to Pierre-Olivier Weill, Joseph Ostroy, Bill Zame, Marek Pycia, Kenneth Mirkin, Cheng Chen and seminar participants at UCLA, the University of Hong Kong, Pennsylvania State University, Peking University, the Chinese University of Hong Kong, the 8th Annual Washington University Graduate Student Conference, the 8th Japan-Taiwan-Hong Kong Contract Theory Conference, Spring 2015 Midwest International Trade and Economic Theory Meetings and 2015 World Congress of the Econometric Society. This work was supported by UCLA Dissertation Year Fellowship. (**JEL classification: D83; L11**)

1 Introduction

Investment incentives and trading outcomes with search frictions have been studied extensively in the holdup and search literature, respectively. In large markets, these two issues are often naturally interrelated: sellers need to decide how much to invest to cut production costs and then start to search for buyers. Investment incentives come from the part of returns on investments that they can reap, which is determined through search and bargaining in the market. At the same time, trading outcomes depend on sellers' investment strategy, as that govern the characteristics of entrants. We show that a lot of equilibrium predictions on investments and trade efficiency are drastically different if we overlook this interdependency.

In this paper, we focus on large markets in which a large number of buyers and sellers search for trading partners. In a large market, the investments are less likely to be relationship-specific or perfectly observable by buyers,¹ which usually hinder efficient investments. We should therefore expect investments to be more efficient in a large market. We showed that it is indeed the case in equilibrium. Surprisingly, we find that although investments are positive, when buyers have all of the bargaining power, social welfare stays the same as if investments were observable (indicating no investment in equilibrium). This result remains true even when investments become socially optimal as search frictions vanish.

The model we propose is a discrete time infinite horizon random search model with exogenous entry and pre-entry investments. At the beginning of each period, there is one unit mass of ex ante identical sellers and buyers entering the market. A buyer entrant demands one unit of output and receives utility $y_0 > 0$ from consuming it. A seller entrant is endowed with the technology to produce one unit of output at cost $x_0 \in (0, y_0)$ and can invest to cut the production cost before entry. We denote the production cost resulting from the socially efficient investment as

¹For example, consider a market that consists of a large number of intermediate suppliers and many retailers. A supplier can invest to locate cheaper producers before searching for retailers. Such investments are not relation-specific, as they increase the surplus from trade with any retailer. Moreover, a retailer is unlikely to be perfectly informed about the cost of a supplier it randomly meets. Another example is the labor market. New graduates, even those with the same academic degree, have heterogeneous costs of labor, partly due to their different levels of costly training.

$x^* \in (0, x_0)$).² After the sunk investments have been made, all agents (entrants and incumbents) randomly form one-buyer-to-one-seller pairs. Within each pair, the buyer makes a take-it-or-leave-it offer without observing the seller's investments. If the offer is accepted, then production takes place and both agents leave the market permanently. Otherwise, the pair is dissolved and both agents search in the next period. At the beginning of the next period, there are new buyers and sellers contemplate entering and the economy repeats the same matching and bargaining process. We assume that agents are impatient and the time between two successive periods is the source of search friction. In the model, agents are referred to as buyers and sellers. However, the agents can have more general roles such as firms and workers, retailers and intermediate suppliers, foreign importers and domestic exporters, etc.

This baseline model produces three main results. First, focusing on the steady-state equilibrium, we show that equilibrium exists and is unique. In equilibrium, ex ante identical seller entrants use mixed strategy when making investments and the resulting distribution of production costs has the support $[x^*, x_0]$. Ex ante identical buyers also use mixed strategy when offering prices and the resulting distribution of prices is non-degenerate.

Second, in spite of the positive amount of investments, the agents' ex ante payoffs and the social welfare equal to the equilibrium values when investments are observable (indicating no investment) and are constant given any search friction. The positive investments create ex post gains. However, the corresponding ex ante gains also depend on the speed of trade. In equilibrium, the ex post gains are completely dissipated by delays in trade given any search friction, due to the possibility of a mismatch between the seller's reserve price and the buyer's offer. This result demonstrates that more efficient investments do not necessarily lead to a higher social welfare.

Third, we investigate how equilibrium objects change as search frictions diminish and what they converge to when search frictions vanish in the limit. First, the investment strategy becomes more efficient in the sense that the proportion of

²The production cost x^* resulting from the socially efficient investments will be defined shortly in the next section.

entrants who invest the socially efficient amount increases to one. In contrast, the distribution of incumbents' pre-entry investments becomes less efficient and converges to a point mass at zero investment. It is because the buyers price more aggressively as search frictions decrease. The relative trading speed between sellers with low and high production costs therefore increases and consequently the market accumulates more underinvested sellers, despite of the fact that more entrants invest efficiently.

In sum, this paper highlights the importance of the interplay between pre-entry investments and trading outcomes. Notice that many of the above results would be drastically different if we separate the two. For instance, it is a robust result that the trading outcome becomes efficient as search frictions vanish if we treat the type distribution of sellers as exogenously given. In addition, more efficient investments in general improve the social welfare if the trading outcomes are held as fixed.

The baseline model is extended along two dimensions to incorporate the possibilities of two-sided investments and two-sided offers, respectively. In the first extension, buyer entrants could also invest before entry to raise their valuations of the good. The results of the baseline model regarding the sellers' investment strategy continue to hold. Moreover, we have a set of new results for the buyer's investment strategy. Most significantly, although the buyers have all of the bargaining power, they still underinvest and mix over an interval of investments given any search friction.

In the second extension, we allow sellers to make take-it-or-leave-it offers occasionally. The social welfare is still completely determined by the lowest investment level, which is now positive as a seller makes offer with a positive probability. Any welfare gain that could be generated from investments above the minimum level is dissipated by delays in trade. In this case, when sellers make offer with strictly positive probability, the minimum investment level and the social welfare converge to the first best outcome as search frictions vanish. The rate of the convergence is higher when sellers make offers more often.

Related Literature

This paper is related to random search models with heterogeneous agents. Albrecht and Vroman (1992) demonstrate that when seller entrants are exogenously

heterogeneous, a single price can never be an equilibrium. The current paper complements their paper by showing that such heterogeneity can emerge endogenously due to diverse prices in equilibrium.

The searching stage of the current model is similar to settings of voluminous works on search and bargaining games with asymmetric information (e.g., Rubinstein and Wolinsky (1990), Satterthwaite and Shneyerov (2007), Shneyerov and Wong (2010a), Lauer mann (2012) (2013), etc). One central topic of the literature is to understand how frictions affect equilibrium efficiency. For instance, Lauer mann (2013) shows that when there exists competitive pressure, equilibrium outcomes converge to perfect competition as search frictions disappear. On the contrary, the social welfare is constant over any search friction in the current setting. The reason is that when the entrants' distribution of production costs is endogenously determined, the equilibrium outcome no longer satisfies weak pairwise efficiency, which is a necessary condition for the convergence as shown in Lauer mann (2013). More detailed discussions can be found in section 4.3. The comparison demonstrates that the equilibrium outcomes are drastically different depending on whether entrants' type distribution is endogenous or not.

In addition, this paper is related to the literature on the hold-up problem. Acemoglu and Shimer (1999) also studies how investment incentives and trading outcomes are jointly determined when there is search friction. They assume that investments are observable, agents use Nash bargaining to split the surplus and entry is endogenous. The different setting compared to us leads to quite different equilibrium outcomes. However, we have one result in common: the conditions for efficient trading when entrants' type distribution is exogenous, the Hosios' condition in theirs and the arbitrarily small search frictions in ours, are no longer sufficient if entrants' characteristics are endogenous.

The incentive of investments is also examined in Gul (2001) within a Coasian setting where a buyer's valuation is determined by his or her unobservable investments. Similar to ours, the equilibrium investment strategy is a mixed strategy and becomes efficient as the time between two rounds shrinks to zero. The difference is that in Gul (2001) there is no bargaining delay in the limit and the social welfare converges to the first best outcome. This difference results from the distinct trading

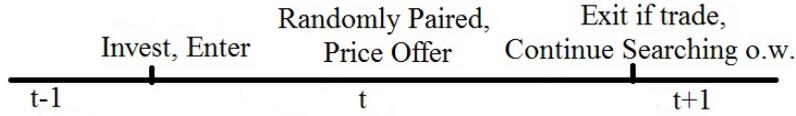


Figure 1: Timeline

environments. We focus on big markets. Because the sellers who invest more trade faster, this selection implies that the incumbents' distribution of ex ante investment is always less efficient than that of entrants. Hence, there are delays in trade even when investments are almost efficient. On the other hand, Gul (2001) focuses on one-to-one trade. Incumbents and entrants have the same distribution of ex ante investment by definition.

The rest of this paper is organized as follows. The model is introduced in Section 2, in which we also solve the first best and observable investment benchmark cases. Section 3 derives the equilibrium conditions and proves the existence and uniqueness of the steady state equilibrium. The equilibrium is characterized in Section 4. Section 5 examines the extension with two-sided investment, and Section 6 considers another extension with two-sided offers. Robustness and other extensions are discussed in Section 7. Finally, Section 8 concludes the paper.

2 Environment and Benchmark Specifications

2.1 Environment

We consider a discrete time infinite horizon random search model with pre-entry investments. Through out this paper, we focus on the steady state equilibrium. Therefore, we omit the time subscript whenever is not confusing. The timeline of this game is illustrated in Figure 1.

Player: The players are sellers and buyers. At the beginning of each period, one unit mass of new sellers and buyers enter the market. A buyer demands one unit of the output and yields utility $y_0 > 0$ from the consumption; a seller entrant is endowed with the technology that produces one unit of output at a cost $x_0 > 0$.

We focus on the “gap” case throughout this paper, i.e., the (minimum) surplus from trade $y_0 - x_0$ is assumed to be strictly positive.

Strategy: Before entering the market, a seller can invest $c(x)$ to cut the production cost to $x \geq 0$. We assume $c(x_0) = c'(x_0) = 0$, $c'(0) < -1$, and for any $x < x_0$, $c(x)$ is of class C^1 , strictly decreasing and strictly convex. All of the entrants then join the incumbents who did not exit in the last period. The market sizes on the two sides are assumed to be the same. In each period, one buyer is randomly matched with one seller and vice versa. The buyer in each pair makes a take-it-or-leave-it offer p , which is a monetary transfer from the buyer to the seller, and the seller decides whether to accept it.

Therefore, a seller’s strategy consists of two components: 1) an investment strategy governed by a CDF $F_e(x)$, which measures the probability that the investment is weakly higher than $c(x)$ and 2) a reserve price $r_S(x)$, which is the lowest price that a seller with cost x is willing to accept. A buyer’s strategy is a price offer governed by a CDF $H(p)$, where $H(p)$ equals the probability of offering a price weakly lower than p .

Preference: If the offer is accepted, then one unit of output is produced and sold, which leaves the seller payoff $p - x$ and the buyer payoff $y_0 - p$. Both agents exit the market permanently. Otherwise, the pair is dissolved and both agents search in the next period.

The time between two successive periods is t and we assume that all agents share the same discount rate r . The discount factor is therefore $\beta = e^{-rt} \in [0, 1)$. We say that the search frictions are small if t is small or equivalently if β is large.

Information: A crucial assumption of this paper is that the buyers have no information about investments. In addition, the matching is anonymous.

2.2 Benchmark Specifications

Benchmark Case 1: The First Best

We first characterize the efficient allocation, which consists of both the efficient investment and trade.

At the search stage, a social planner would find it optimal to always conduct

trade between any two agents given any cost distribution, as the surplus from trade is always positive and postponing trading is costly due to discounting.

Given that trades take place immediately at the search stage, if a seller invests to cut his or her cost to x , then he or she increases the social welfare by $x_0 - x$ with an investment cost $c(x)$. A social planner thus chooses x^* implicitly defined by $c'(x^*) = -1$ so that the marginal cost of investment equals the marginal benefit. From the assumptions on $c(x)$, it is easy to verify that $x^* \in (0, x_0)$.

Benchmark Case 2: Observable Investments

Consider the situation in which investments are observable. Following the same argument used in Diamond (1971), as the buyers have all of the bargaining power, a seller receives zero search stage payoff regardless of his or her production cost. The reasoning is as follows. The buyer in the current match and buyers in all future matches will offer exactly the seller's production cost plus the discounted continuation payoff, as the buyers can observe the seller's production cost. With the discount factor being strictly less than one, this infinitely repeated discounting drives the search stage payoff down to zero.

Therefore, in the unique equilibrium, no seller invests even if the investment is not relation-specific and all of the buyers offer a price x_0 . Investments are inefficient and trades are efficient.

This analysis shows why unobservability is necessary to restore investment incentives. When buyers cannot observe (or cannot perfectly infer in equilibrium) investments, a seller may receive a price offer higher than his or her reservation price. Such a possibility creates rents for sellers. We illustrate this intuition in detail in the next section.

3 The Steady State Equilibrium

Let us now solve the steady state equilibrium in the decentralized market. A steady state equilibrium consists of four elements: a seller's investment strategy when he or she is an entrant and reserve price function when he or she is an incumbent, a buyer's price offer distribution and a distribution of incumbents' production costs. Following the literature, we will call the last distribution as stationary cost distribu-

tion.

3.1 The Seller's Problem

At the search stage, a seller with a production cost x chooses the lowest price he or she is willing to accept, i.e., the reserve price $r_S(x)$, to maximize his or her search stage payoff $U(x)$. Given a price offer distribution $H(p)$, the seller's trading probability is $1 - H(r_S(x)) + Pr(p = r_S(x))$, which is decreasing in $r_S(x)$. Here, $Pr(p = r_S(x))$ is the probability of a price offer that equals $r_S(x)$ according to $H(p)$. The maximization problem of a type x seller can be summarized as follows:

$$U(x) = \max_r \{ (E(p | p \geq r) - x)(1 - H(r) + Pr(p = r)) + (H(r) - Pr(p = r))\beta U(x) \} \quad (1)$$

Solving the above problem, the reserve price $r_S(x)$ is:

$$r_S(x) = x + \beta U(x) \quad (2)$$

Intuitively, a seller is willing to accept any price that is high enough to cover his or her opportunity cost of trading, i.e., the production cost x plus the discounted continuation payoff.

A more efficient seller should have a higher search stage payoff $U(x)$, as investments are costly. This more efficient seller is willing to accept lower price offers in equilibrium, as the opportunity cost of delay is higher. Moreover, the least efficient seller should receive zero search stage payoff. The reason is that no buyer in equilibrium would offer prices higher than his or her reserve price, which implies zero payoff following the insights in Diamond's Paradox. Denote the highest production cost on the support as \bar{x} . The following lemma confirms the preceding conjectures.

Lemma 1. *In any steady state equilibrium, $U(x)$ is strictly decreasing and continuous in x , with $U(\bar{x}) = 0$, and $r_S(x)$ is strictly increasing and continuous in x .*

Unless otherwise mentioned, all of the proofs are gathered in the Appendices.

Lemma 1 implies that $\hat{x}(p)$, the inverse function of $r_S(x)$, is well defined, continuous and strictly increasing. Function $\hat{x}(p)$ specifies the highest type of a seller who is willing to accept a price p .

Lemma 1 also implies that the highest cost on the support equals the initial cost, i.e. $\bar{x} = x_0$. A seller with the highest cost is fully extracted in the search stage and therefore has no incentive to invest ex ante. In other words, the holdup problem remains for the least efficient sellers, although their investments are unobservable and they can search for other buyers.

Corollary 1. *In any steady state equilibrium, the least efficient sellers invest zero, $\bar{x} = x_0$.*

The ex ante payoff of the least efficient sellers therefore equals $U(\bar{x}) - c(x_0) = 0$. In equilibrium, ex ante identical sellers must be indifferent over any x on the support of F_e , which governs the investment strategy, and weakly prefer these x to any other x that is not on the support. As can be seen from (1), the search stage payoff $U(x)$ depends on the price offer distribution $H(p)$. In equilibrium, $H(p)$ must be such that

$$\begin{aligned} U(x) - c(x) &= 0, \text{ for any } x \text{ on the support of } F_e(x) & (3) \\ U(x) - c(x) &\leq 0, \text{ for any } x \text{ not on the support of } F_e(x) \end{aligned}$$

3.2 The Buyer's Problem

A buyer chooses what price to offer. We know from the previous section that any seller with a cost lower than $\hat{x}(p)$ is willing to accept p . We use $F(x)$ to denote the CDF of the stationary cost distribution, which equals the probability that the production cost of a randomly drawn incumbent is weakly lower than x . The probability of trade therefore equals $F(\hat{x}(p))$ for price p .

In equilibrium, $F(x)$ must be such that it makes a buyer indifferent over any p on the support of $H(p)$. We use π to denote the equilibrium payoff of a buyer. Then any p on the support solves the following maximization problem:

$$\pi = \max_p \{(y_0 - p)F(\hat{x}(p)) + (1 - F(\hat{x}(p)))\beta\pi\} \quad (4)$$

3.3 The Seller's Investment Strategy

The last piece of the model is the distribution of seller entrants' production cost, which is also the seller's investment strategy.³ In a steady state equilibrium, the measure of outflow of any type must equal the measure of inflow of the same type to preserve the stationary distribution over time. A seller with a cost x leaves the market if he or she receives an offer that is weakly higher than $r_S(x)$ (which happens with a probability $1 - H(r_S(x))$).⁴ Meanwhile, the measure of entrants with a cost lower than x is $F_e(x)$. Denote the lowest production cost on the support as \underline{x} . The steady state equilibrium requires that for any x on the support,

$$F_e(x) = \frac{F(x) - \int_{\underline{x}}^x H(r_S(\tilde{x}))dF(\tilde{x})}{1 - \int_{\underline{x}}^{x_0} H(r_S(\tilde{x}))dF(\tilde{x})} \quad (5)$$

3.4 Equilibrium Existence and Uniqueness

Let us first summarize the dynamics of a steady state equilibrium. At any period, the stationary cost distribution $F(x)$ is such that it keeps buyers indifferent to price offers. The players' trading strategy then determines the cost distribution of those who exit. At the beginning of the next period, the new generation of seller entrants, who are indifferent to these investment amounts given the price distribution $H(p)$, choose the amount of investments so that they exactly replace those who exit. This way, the stationary cost distribution is preserved over time.

Proposition 1. *In any steady state equilibrium, the price offer distribution $H(p)$, the seller's investment strategy $F_e(x)$ and the stationary cost distribution $F(x)$ have the following properties:*

1. $F(x)$ and $F_e(x)$ have support $[x^*, x_0]$ with the unique point mass at x^* .
2. $H(p)$ has support $[r_S(x^*), r_S(x_0)]$ and is atomless;

³As there is one unit mass of entrants, we obtain the equivalence between the distribution of entrants' costs and the investment strategy when we abuse the law of large numbers as usual.

⁴To be more precise, the probability of receiving an offer weakly higher than $r_S(x)$ should be $1 - H(r_S(x)) + Pr(p = r_S(x))$. However, as is proved in the next section, there is no mass point on the support of $H(p)$. To simplify the notation, we write down the probability here as if there is no mass point.

Although all agents are identical ex ante, proposition 1 shows that the stationary price offer distribution and investment strategy are non-degenerate. The unobservability of the investments is the key behind this result. Suppose otherwise, i.e., that all buyers offer the same price. The sellers become residual claimants facing the single price, as the unobservability of investments prevents any further exploitation. They either invest efficiently or invest zero. If the price is not high enough to cover the investment cost plus the production cost ($p < c(x^*) + x^*$), then the sellers invest zero. In addition, because $c(x^*) + x^* < x_0$, the price is lower than the endowed production cost. As a result, the sellers reject the offer after entering the market. This leaves the buyers with zero profit. Therefore, to earn a positive profit, the buyers must offer a price \hat{p} higher than $c(x^*) + x^*$ so that the sellers invest efficiently ex ante and agree to accept the price. However, given that all of the other buyers offer the price \hat{p} , a buyer would find it profitable to deviate to a slightly lower price. Due to discounting, the matched seller is still willing to accept the lower price. Hence, a single price can never be an equilibrium. To support such dispersed price distribution, the investment strategy also has to be non-degenerate.

We may suspect a gap on the support of the price and stationary cost distributions. For instance, it may be the case that two prices p_1 and p_2 are on the support, but any price between p_1 and p_2 is not because the sellers with costs between $\hat{x}(p_1)$ and $\hat{x}(p_2)$ are not in the market and hence the buyers do not gain from offering their reserve prices. Meanwhile, no sellers choose to cut cost to $x \in (\hat{x}(p_1), \hat{x}(p_2))$ because their reserve prices are not offered. This intuition unfortunately neglects the indifference condition: a seller must be indifferent between $\hat{x}(p_1)$ and $\hat{x}(p_2)$ and weakly prefer them to any $x \in (\hat{x}(p_1), \hat{x}(p_2))$. As no price between p_1 and p_2 is offered, $U(x)$ increases linearly on the interval $(\hat{x}(p_1), \hat{x}(p_2))$. However, the investment cost function $c(x)$ is strictly convex. Hence, the indifference condition could never hold if there were a gap on the support.

Proposition 1 also claims that there is no mass point on the price distribution. Any mass point results in a jump in the probability of trade, which in turn leads to a kink in $U(x)$. However, the investment cost function $c(x)$ is of class C^1 . This again contradicts the indifference condition.⁵

⁵Of course, the preceding reasoning is based on the assumption that the investment cost function

The lowest production cost on the market is the socially efficient cost x^* according to Proposition 1. The seller with the lowest cost trades immediately with a probability of one: any price offer in the market is weakly higher than his or her reserve price. As the seller's investments are unobservable to the buyer, he or she becomes the residual claimant and invests efficiently. In addition, there is a mass point at x^* in both $F_e(x)$ and $F(x)$, as a buyer who offers $r_S(x^*)$ can only trade with sellers x^* and must get strictly positive equilibrium payoff. Moreover, there is no other mass point, as any of such point would lead to a jump in the buyer's payoff as a function of p . This contradicts the buyer's indifference condition.

The preceding results and intuitions hold even when $\beta = 0$, in which case the buyer in each pair is a monopolist. The monopolist uses mixed pricing strategy because the stationary cost distribution is adjusted so that the demand function is unit elastic at any price on the support.

In the rest of this section, we solve $H(p)$ and $F(x)$ and then show the existence and uniqueness of the steady state equilibrium. $H(p)$ can be solved from the envelope condition of $U(x)$. It is legitimate to take the derivative of $U(x)$ because we have proved that the support is an interval and that $U(x) - c(x) = 0$ for any x on the support. The envelope condition is

$$U'(x) = -(1 - H(r_S(x))) + H(r_S(x))\beta U'(x)$$

Using the equilibrium restriction that $U'(x) = c'(x)$, $H(p)$ can be solved:

$$H(p) = \frac{1 + c'(\hat{x}(p))}{1 + \beta c'(\hat{x}(p))} \quad (6)$$

$F(x)$ is solved from the buyer's indifference condition. If a buyer offers the highest reserve price $r_S(x_0) = x_0 + \beta U(x_0) = x_0$, he or she can trade with a probability of one. Therefore, $\pi = y_0 - x_0$.

Any other price on the support must yield the same expected profit. In other

 $c(x)$ is of class C^1 , continuous and strictly convex. If $c(x)$ is not so well behaved, then some properties of the price distribution change accordingly.

words,

$$(y_0 - p)F(\hat{x}(p)) + [1 - F(\hat{x}(p))]\beta\pi = y_0 - x_0$$

Therefore, the stationary cost distribution $F(x)$ can be calculated as follows:

$$F(x) = \begin{cases} 0, & \text{if } x \in (-\infty, x^*), \\ \frac{y_0 - \beta\pi - x_0}{y_0 - \beta\pi - x - \beta c(x)}, & \text{if } x \in [x^*, x_0], \\ 1, & \text{if } x \in (x_0, +\infty). \end{cases} \quad (7)$$

We have shown that $F(x)$ and $H(p)$ exist and are unique. Thus, the investment strategy $F_e(x)$ also exists and is uniquely determined by (5).

Proposition 2. *The steady state equilibrium exists and is unique.*

4 Equilibrium Characterization

4.1 Investment Strategy and Stationary Cost Distribution

There are two distributions of sellers' production costs. One is the investment strategy $F_e(x)$, which is the cost distribution of entrants. The other is the stationary cost distribution $F(x)$, which is the cost distribution of incumbents. We have already shown that the per-period trading probability increases in the investment level of a seller, or equivalently decreases in the production cost. Therefore, a more efficient seller trades and exits the market more rapidly. This in turn implies that the investment strategy $F_e(x)$ is always more efficient than $F(x)$.

Proposition 3. *The stationary cost distribution $F(x)$ has first order stochastic dominance over the investment strategy $F_e(x)$.*

4.2 Constant Payoffs and Social Welfare

As shown in the previous section, sellers always invest with a positive probability given any search friction. One may expect the social welfare to be higher than that

obtained in the benchmark case with observable investments, where sellers have no incentive to invest. However, this is not the case as shown in the following theorem.

Theorem 1. *For any $\beta \in [0, 1)$, the seller's ex ante payoff is 0, and the buyer's ex ante payoff equals the social welfare, which is $y_0 - x_0$.*

We can easily verify that the agents' payoffs and social welfare are the same when investments are observable and perfectly unobservable. Although the unobservability provides incentive to invest, it also causes trading inefficiency. The welfare gain generated from investments can be realized fully only if the seller and buyer entrants agree to trade immediately after entry. However, this is impossible given the presence of information and search frictions. Due to the unobservability, both the cost and the price distributions are non-degenerate. Socially efficient trades are therefore conducted only probabilistically. In other words, there is an expected delay in trade for any buyer offering lower than $r_S(x_0)$. The welfare loss due to the delay in trade exactly offsets the welfare gain from investments.

The social welfare remains constant over the search frictions. As the social welfare depends on both the investment and trade efficiency, the constant welfare result as search frictions diminish may arise from the constant investment and trade efficiency, from more efficient investments and less efficient trade or vice versa. To determine which is the case, in the next section we investigate the change in investment strategy and efficiency of trade as we vary the search frictions.

4.3 Comparative Statics and the Limiting Results

In this paper, there is search friction in the sense that an agent needs to wait time t to meet the next potential partner. In this section, we illustrate how equilibrium outcomes vary in search frictions, or equivalently, in t and β .

Consider a seller with cost $x \in (x^*, x_0)$. As meetings become more frequent, the seller is more likely to trade per unit of time if the price distribution remains constant. Consequently, the new marginal benefit of investment, which strictly increases in the probability of trade per unit of time, is strictly higher. Therefore, to keep a seller indifferent across investments when the search frictions decrease,

the buyers must price more aggressively. That is, the per-period trading probability $1 - H(r_S(x))$ must strictly decrease in β . Indeed,

$$\frac{\partial(1 - H(r_S(x)))}{\partial\beta} = \frac{(1 + c'(x))c'(x)}{(1 + \beta c'(x))^2} < 0, \text{ for any } x \in (x^*, x_0)$$

As search frictions vanish, the probability of trade per-period $1 - H(r_S(x))$ must converge to zero for any $x \in (x^*, x_0)$. Equivalently, buyers must price extremely aggressively in the limit, i.e., the price offer distribution must converge in distribution to a point mass at $r_S(x^*)$. Otherwise, any seller trades almost certainly within any small amount of time. The marginal benefit of investment therefore becomes one and a seller cannot be indifferent across investment levels. We can also mathematically verify this intuition from (6) by taking the limit $\beta \rightarrow 1$.

In addition, the stationary cost distribution $F(x)$ becomes less efficient as β increases. If $F(x)$ stays constant when meetings become more frequent, a buyer who is originally indifferent over price offers strictly prefers to offer the lowest price $r_S(x^*)$. Therefore, the per-period trading probability of a buyer offering $p < r_S(x_0)$ must decrease in β . That is, the stationary cost distribution $F(x)$ with a larger β has first order stochastic dominance over an $F(x)$ with a smaller β . Indeed,

$$\frac{\partial F(x)}{\partial\beta} = \frac{(y_0 - x_0)(x - x_0 + c(x))}{[y_0 - \beta\pi - x - \beta c(x)]^2} < 0$$

In the limit, we can verify that $F(x)$ converges in distribution to a point mass at x_0 from (7). That is, almost all of the incumbents are composed of sellers who invested zero ex ante. This result is shown graphically in Figure 2. If $F(x)$ is not converging to 0 in the limit for some $x < x_0$, then a buyer offering $r_S(x)$ has strictly positive trading probability per-period and trades immediately. Hence, a buyer who offers $r_S(x_0)$ would find it optimal to lower the price to $r_S(x)$ without changing the trading speed, leading to a contradiction.

In the steady state, the cost distribution of entrants is the same as that of exits to preserve the stationary cost distribution. As the cost distribution of incumbents becomes less efficient as β increases, one may suspect the distribution of exits and hence the investment strategy $F_e(x)$ to be less efficient as well.

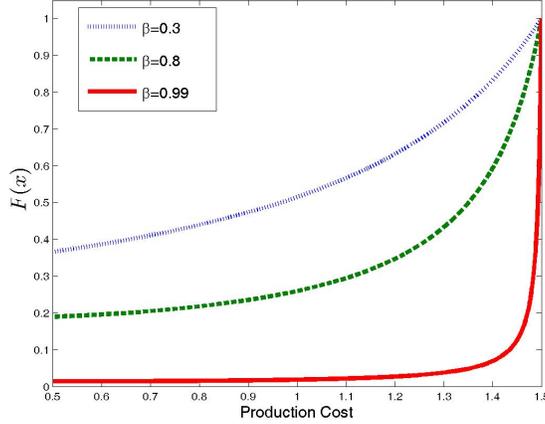


Figure 2: Stationary Seller Type Distribution
(In this example: $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$.)

It turns out that the conjecture is not quite right. We find that the point mass $F_e(x^*)$ strictly increases in β , i.e., an entrant is more likely to invest the socially efficient amount. In the limit, $F_e(x^*)$ increases to one and hence the investment strategy becomes efficient. We have already shown that $F_e(x)$ is always more efficient than $F(x)$. Our limiting result further shows that the two distributions converge to two polar points, respectively. The investment strategy with different discount factors is plotted in Figure 3.

To understand this result, note that a seller who invests efficiently always exits the market immediately independent of β . In addition, we have three observations from previous analysis: as β increases, 1) a larger proportion of the incumbents underinvest ex ante; 2) the per-period trading probabilities of underinvested sellers strictly decrease; and 3) the average cost of underinvested sellers strictly increases, which implies that a seller with the average cost is less likely to trade. The first effect raises the proportion of exits who underinvest. The remaining two effects explain why the proportion changes in the opposite direction: any underinvested seller is less likely to trade and the composition of the underinvested sellers becomes less efficient. The latter two effects dominate the former.

We summarize the preceding comparative statics and limiting results in the following theorem.

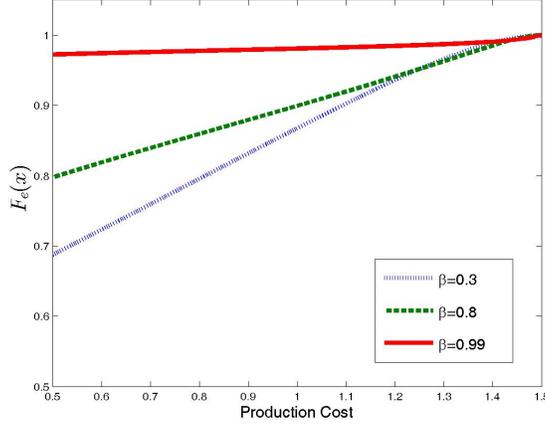


Figure 3: Seller's Investment Strategy
(In this example: $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$.)

Theorem 2. As β increases to one, in the steady state equilibrium

1. $H(r_S(x))$ strictly increases for any $x \in (x^*, x_0)$ and converges in distribution to a point mass at $r_S(x^*)$;
2. $F(x)$ strictly decreases for any $x \in [x^*, x_0)$ and converges in distribution to a point mass at x_0 ; and
3. $F_e(x^*)$ strictly increases and $F_e(x)$ converges in distribution to a point mass at x^* .

From this theorem, we can better understand the constant social welfare result. As β increases, more of the new entrants invest efficiently, and this may generate additional social welfare if the trading efficiency remains constant. Unfortunately, trades become less efficient at the same time. As β increases, the stationary cost distribution concentrates more on higher costs. Consequently, it takes more periods for a buyer offering a given seller's reserve price to trade.

This is very different from the result in Lauermaann (2013): when entrant's type distribution is exogenously given, the sequence of equilibrium converges to the perfect competition under the same bargaining protocol and information structure in the current model. A necessary condition for such convergence is weak pairwise

efficiency, that is, $U(x) + \pi \geq y - x$ for any $x \in (x^*, x_0)$. This is equivalent to $U(x) \geq x_0 - x$, which fails to hold here. This means that whether weak pairwise efficiency is violated or not depends on whether the entrant's type distribution is endogenously determined or not. When it is exogenous, sellers of types $x \in (x^*, x_0)$ who trade with probability less than 1 must comprise strictly positive proportion of incumbents. As a result, a buyer meets a seller of type $x \in (x^*, x_0)$ with strictly positive probability. However, when the type distribution is endogenously determined, to ensure that buyers are indifferent between price offers, the proportion of incumbents with production costs $x < x_0$ must converge to zero in the limit. In other words, although sellers of types $x \in (x^*, x_0)$ still trade with probability less than 1, a buyer meets these sellers with probability arbitrarily close to zero in the limit. In this case, the gains from trade $y - x - U(x) - \pi$ can be strictly positive in the limit, which violates weak pairwise efficiency.⁶

4.4 Convergence from the initial time to the steady state

So far we have focused on the steady state equilibrium. In this section, we investigate the existence of a path starting from date zero and converging to the steady state, such that the equilibrium properties in the steady state are preserved along the path.

To construct such a path, we assume that agents in any cohort (correctly) expect to receive the same search stage payoffs as in the steady state equilibrium. In period $t = 1$, we let the seller entrants to invest according to $F(x)$ and the buyers to price according to $H(p)$. Seller entrants obey the order because they are indifferent to any investment level on the support given their expected search stage payoffs. Buyers also obey, as the cost distribution of incumbents at $t = 1$ is $F(x)$. At the end of date $t = 1$, only some of the agents (denoted by $e_1 < 1$) trade and exit, and the cost distribution of exiting sellers is exactly $F_e(x)$.

At the beginning of date $t = 2$, the new cohort of sellers are again indifferent over investment levels. Let measure e_1 of them invest according to $F_e(x)$ and

⁶To be more precise, Lauer mann(2013) shows that the product of the trading probabilities and gains from trade converges to zero in the limit. Given that the trading probabilities converge to zero when entrants' type distribution is endogenous, the gains from trade can be positive.

measure $1 - e_1$ of them invest according to $F(x)$. This way, the incumbents' cost distribution at $t = 2$ remains $F(x)$. The price distribution of buyers is therefore the same. Furthermore, the size of exits at $t = 2$, e_2 , remains smaller than one.

In general, at any date along the path, the size of the exits is always smaller than the size of the entrants. Therefore, the investment distribution of entrants in any cohort can be constructed as previously specified to preserve $F(x)$. The size of the market will eventually grow to the size in steady state and the size of the exits will be equal to that of the entrants. In the end the equilibrium converges to the steady state.

As the preceding construction shows, the agents' ex ante payoffs, social welfare, incumbents' cost distribution and price distribution are constant along the path.

Proposition 4. *There exists an equilibrium path starting from the initial time and converging to the steady state such that*

1. *at any point along the path, the incumbents' cost distribution is $F(x)$ as specified in (7) and the price distribution is $H(p)$ as specified in (6);*
2. *within any cohort, sellers' ex ante payoff is zero, buyers' ex-ante payoff is $y_0 - x_0$ and the social welfare is $y_0 - x_0$; and*
3. *the investment strategy becomes more efficient as time increases.*

The rest of the paper extends the baseline model along two directions. Section 5 considers the situation in which a buyer may invest to raise the valuation. Section 6 examines a two-sided offering extension in which a seller makes a take-it-or-leave-it offer with a positive probability.

5 Two-Sided Investments

In many situations, buyers can also invest before searching for sellers. For instance, a firm can invest in technology to raise the output per unit of labor. The main message in this extension is that the buyers in this environment underinvest even if they have all of the bargaining power.

We assume that before entering, a buyer can increase the valuation from y_0 to y with investments $e(y)$, where $e(y_0) = e'(y_0) = 0$ and $e(y)$ is strictly increasing and strictly convex for any $y > y_0$. The surplus from trade between x and y is $y - x$. This assumption excludes any complementarity between investments, which greatly simplifies the analysis⁷. Finally, the observability of buyers' investments may be arbitrary, as there is no complementarity by assumption and buyers have all of the bargaining power.

Consider the benchmark cases of a) the first best and b) observable investments. In the first best benchmark case, as there is no complementarity, the social planner commends all agents to trade upon their first meetings. Given this, the planner asks all sellers to reduce the production cost to x^* and all buyers to raise the value to y^* , where y^* is uniquely determined by $e'(y^*) = 1$.

In the benchmark case with observable investments, all sellers invest zero and all buyers raise the valuation to y^* and offer a price x_0 . In equilibrium, a seller receives a payoff of zero and a buyer receives a payoff of $y^* - x_0 - e(y^*)$, which is also the social welfare.

In the rest of this section, we derive the optimality conditions of the steady state equilibrium and then characterize the equilibrium.

5.1 The Steady State Equilibrium

Due to the lack of complementarity, sellers cannot benefit from the buyers' investments directly and care only about the price distribution. Therefore, the seller's problem is exactly the same as before and the previous equilibrium conditions for sellers continue to hold.

We therefore focus on the buyer's problem in the rest of this section. A buyer's strategy consists of his or her investment strategy $G_e(y)$ and the price offer $p(y)$.

⁷In the case with one-sided investment, we can make this assumption without a loss of generality, as all of the buyers are identical. In the two-sided investment case, however, this assumption rules out some interesting scenarios. For instance, the surplus from trade could be supermodular, in which case a higher seller's investment level leads to a larger marginal benefit of the buyer's investment. We will leave the analysis for more general cases as future work.

$p(y)$ maximizes the search stage payoff $\Pi(y)$,

$$\Pi(y) = \max_p \{(y - p)F(\hat{x}(p)) + [1 - F(\hat{x}(p))]\beta\Pi(y)\} \quad (8)$$

Moreover, the following indifference conditions must hold.

$$\Pi(y) - e(y) = \pi \geq 0, \text{ for any } y \text{ on the support of } G_e(y) \quad (9)$$

$$\Pi(y) - e(y) \leq \pi, \text{ for any } y \text{ not on the support of } G_e(y)$$

Here π is the buyer's ex ante payoff.

Lemma 2. *In any steady state equilibrium with two-sided investments, $p(y)$ is single-valued and increases in y for any y on the support of $G_e(y)$.*

Lemma 2 shows that the price offer increases in buyer's type. The waiting cost of a buyer with a higher valuation is larger. The buyer is willing to offer a higher price to ensure a higher trading probability. Consequently, the buyer with the highest valuation on the support, denoted as \bar{y} , offers the highest price, which equals the reserve price of a type x_0 seller, i.e., $p(\bar{y}) = x_0$ and $F(\hat{x}(p(\bar{y}))) = 1$. Therefore, $\Pi(\bar{y}) = \bar{y} - x_0$ and $\pi = \Pi(\bar{y}) - e(\bar{y}) = \bar{y} - x_0 - e(\bar{y}) > 0$.

Lemma 2 also demonstrates that given his or her ex ante investments, a buyer will never play a mixed pricing strategy at the search stage. Suppose that there are two buyers investing the same $e(y)$ but offering different prices $p_1 > p_2$. Given the non-degenerate production cost distribution, buyer 1 trades faster in expectation and therefore has a larger marginal benefit of investment. This contradicts the fact that both buyers invested the same amount.

More importantly, the pure pricing strategy implies that although the buyers have all of the bargaining power, they adopt a mixed investment strategy and hence underinvest with a strictly positive probability.

Proposition 5. *In any steady state equilibrium with two-sided investments, the seller's investment strategy $F_e(x)$, stationary cost distribution $F(x)$, buyer's investment strategy $G_e(y)$ and stationary valuation distribution $G(y)$ have the following properties:*

1. $F_e(x)$ and $F(x)$ have support $[x^*, x_0]$ with the unique point mass at x^* ; and
2. $G_e(y)$ and $G(y)$ have support $[\underline{y}, y^*]$ and are atomless, where \underline{y} is uniquely determined by

$$y^* - x_0 - e(y^*) = [\underline{y} - x^* - \beta c(x^*)]e'(\underline{y}) - e(\underline{y}) \quad (10)$$

Finally, the steady state condition must hold for buyers. That is, the investment strategy $G_e(y)$ must equal the valuation distribution of the buyers who exit the market. A buyer exits when his or her offer is accepted, which happens with a probability of $F(\hat{x}(p(y)))$. Combined with $G(y)$, the distribution of exits, which also equals the distribution of entrant is determined.

$$G_e(y) = \frac{\int_{\underline{y}}^y F(\hat{x}(p(\tilde{y})))dG(\tilde{y})}{\int_{\underline{y}}^{y^*} F(\hat{x}(p(\tilde{y})))dG(\tilde{y})} \quad (11)$$

We are now ready to solve the equilibrium. The convex supports and the indifference conditions imply that both $U(x)$ and $\Pi(y)$ are differentiable. We can therefore use the envelope conditions to solve for the stationary distributions and the price offer function $p(y)$. The derivation also proves the existence and uniqueness of the steady state equilibrium. We summarize the results in the following lemma.

Lemma 3. *The steady state equilibrium with two-sided investments exists and is unique. The stationary cost distribution CDF $F(x)$ is defined by (13), the sellers' investment strategy CDF $F_e(x)$ is defined by (5), the reserve price $r_S(x)$ is defined by (2), the stationary valuation distribution CDF $G(y)$ is defined by (14), the buyers' investment strategy CDF is defined by (11) and the price offer $p(y)$ is defined by (12).*

$$p(y) = y - \frac{e(y) + y^* - x_0 - e(y^*)}{e'(y)} \quad (12)$$

$$F(x) = \begin{cases} 0, & \text{if } x \in (-\infty, x^*), \\ \frac{(1-\beta)e'(\hat{y}(r_S(x)))}{1-\beta e'(\hat{y}(r_S(x)))}, & \text{if } x \in [x^*, x_0], \text{ where } \hat{y}(p) \text{ is the inverse of } p(y) \\ 1, & \text{if } x \in (x_0, +\infty). \end{cases} \quad (13)$$

$$G(y) = \begin{cases} 0, & \text{if } y \in (-\infty, \underline{y}) \\ \frac{1+c'(\hat{x}(p(y)))}{1+\beta c'(\hat{x}(p(y)))}, & \text{if } y \in [\underline{y}, y^*], \\ 1, & \text{if } y \in (y^*, +\infty). \end{cases} \quad (14)$$

In the baseline model, we have demonstrated that the equilibrium payoffs and social welfare are the same as if investments were observable. This result still holds in the two-sided investments extension.

Theorem 3. *In the steady state equilibrium with two-sided investments, the seller's ex ante payoff equals zero, the buyer's ex ante payoff and the social welfare equals $y^* - x_0 - e(y^*)$.*

5.2 Comparative Statics and the Limiting Case

The comparative statics and limiting results for $H(p)$, $F(x)$ and $F_e(x)$ in theorem 2 can be extended here. We do not repeat the results but instead devote this section to the comparative statics exercise on the buyers' investment strategy $G_e(y)$ and stationary valuation distribution $G(y)$.

First, the lower bound \underline{y} strictly increases in β . To understand this result, we know that \underline{y} offers the lowest reserve price $x^* + \beta c(x^*)$, which strictly increases in β . Meanwhile, the trading probability is strictly lower, as $F(x^*)$ strictly decreases in β . Therefore, \underline{y} must strictly increase in β to keep the buyer's ex ante payoff constant ($\pi = y^* - x_0 - e(y^*)$). Furthermore, the limit of \underline{y} as $\beta \rightarrow 1$ is strictly less than y^* . In other words, the investment strategy is non-degenerate even when the search frictions vanish.

The stationary valuation distribution $G(y)$ also adjusts as β changes. We know from the baseline model that the price distribution concentrates more on lower prices as β increases. Given that $p(y)$ strictly increases in y , it must be the case that $G(y)$ concentrates more on lower valuations. $G(y)$ indeed converges in distribution to a point mass at \underline{y} in the limit. Figure 4 graphically shows the preceding

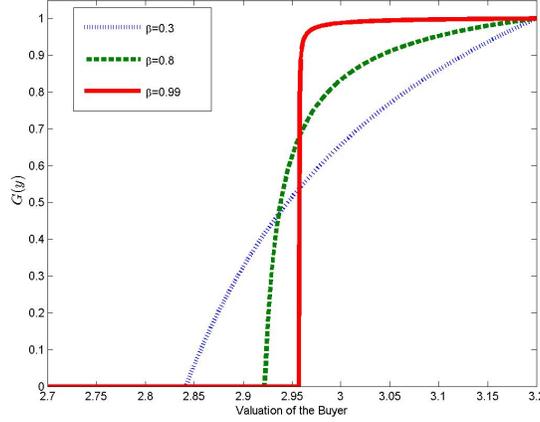


Figure 4: Stationary Buyer Type Distribution

(In this example: $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$, $e(y) = \frac{1}{2}(y - y_0)^2$.)

two results.

The buyers' investment strategy $G_e(y)$ also converges in distribution to a point mass at \underline{y} as β approaches one. We know that almost all of the buyers offer the lowest price in the limit. Therefore, almost all of the transaction prices equal $p(\underline{y})$. Hence, the entrants who replace these exits in the steady state consist almost entirely of type \underline{y} buyers. The investment strategy with the same set of parameters is plotted in Figure 5.⁸

The preceding discussions are summarized in proposition (6).

Proposition 6. *In the steady state equilibrium with two-sided investments,*

1. *as β increases to one, (i) the lowest valuation \underline{y} strictly increases and (ii) a buyer with a valuation that equals the $t * 100$ th percentile of $G(y)$ offers the reserve price of a more efficient seller, for any $t \in (0, 1)$; and*
2. *as $\beta \rightarrow 1$, (i) \underline{y} is still bounded away from y^* , i.e., $\lim_{\beta \rightarrow 1} \underline{y} < y^*$, and (ii) $G_e(y)$ and $G(y)$ converge in distribution to a point mass at \underline{y} .*

⁸To clarify, although the investment strategy concentrates more on lower valuations, we cannot conclude that the investment strategy becomes less efficient because \underline{y} strictly increases in β .

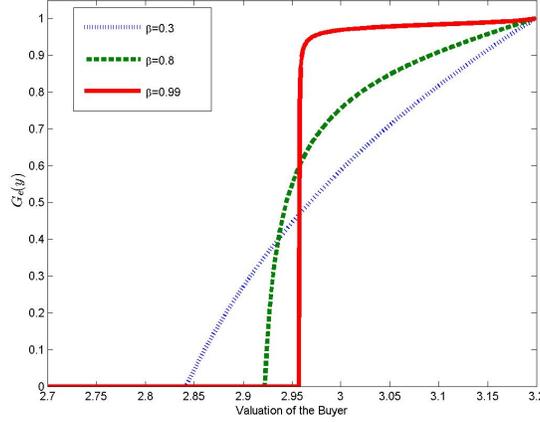


Figure 5: Buyer's Investment Strategy

(In this example: $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$, $e(y) = \frac{1}{2}(y - y_0)^2$.)

6 Two-Sided Offers

In this section, we consider the extension where sellers also have opportunities to make offers. In each meeting, assume nature randomly selects the seller to make a take-it-or-leave-it offer with a probability $\alpha > 0$ and selects the buyer with the complementary probability. A seller's strategy then also includes a price offer $p_S(x)$, and a buyer's strategy also includes a reserve price r_R . It is easy to see that $r_R = y_0 - \beta\pi$. Therefore, all sellers will propose $p_S(x) = r_R$ if $r_R - x$ is weakly higher than $\beta U(x)$.

6.1 Benchmark Case: Observable Investments

As a benchmark, let us first characterize the steady state equilibrium when investments are observable. In this case, there are two candidates of the optimal investments levels: 0 and $c(\bar{x})$, where

$$\bar{x} = \operatorname{argmax}_x \left\{ \frac{\alpha(y_0 - x - \beta\pi)}{1 - \beta(1 - \alpha)} - c(x) \right\}$$

Or equivalently, \bar{x} is uniquely defined by

$$c'(\bar{x}) = \frac{-\alpha}{1 - \beta(1 - \alpha)} \quad (15)$$

Lemma 4. *In any steady state equilibrium with two-sided offers and observable investments, all of the seller entrants invest $c(\bar{x})$. Either all the incumbents invested $c(\bar{x})$ and trade immediately, or some of them invested 0 and never trade.*

Therefore, depending on the parameters, one of the following two equilibria arises. In the first type of equilibrium (named active equilibrium from now on), all the incumbents invested $c(\bar{x})$ and all of the agents trade without delay. To sustain the active equilibrium, the search stage payoff, which depends on the bargaining power α and the shape of $c(x)$, must be larger than the investment cost.

When this condition fails to hold, we have the second type of equilibrium (named partial-active equilibrium from now on), in which some of the incumbents invested zero and never trade. The remaining incumbents invested $c(\bar{x})$, trade immediately and are replaced by new entrants in the next period. Then seller entrants must be indifferent between investing zero and $c(\bar{x})$.⁹

Lemma 5. *The steady state equilibrium with two-sided offers and observable investments takes one of the following two forms.*

Active Equilibrium: *If $\alpha(y_0 - \bar{x}) \geq c(\bar{x})$, then both $F(x)$ and $F_e(x)$ are a point mass at \bar{x} . The seller's ex ante payoff is $\alpha(y_0 - \bar{x}) - c(\bar{x})$ and the buyer's ex ante payoff is $(1 - \alpha)(y_0 - \bar{x})$.*

Partial-Active Equilibrium: *If $\alpha(y_0 - \bar{x}) < c(\bar{x})$, then $F_e(x)$ is a point mass at \bar{x} and $F(x)$ is a two-point distribution with $F(\bar{x}) = \frac{\alpha(y_0 - \bar{x})}{\beta(1 - \alpha)c(\bar{x})} - \frac{1 - \beta(1 - \alpha)}{\beta(1 - \alpha)}$ and $F(x_0) = 1 - F(\bar{x})$. The seller's ex ante payoff is zero and the buyer's ex ante payoff is $\frac{\alpha(y_0 - \bar{x}) - (1 + \alpha\beta - \beta)c(\bar{x})}{\alpha\beta}$.*

From (15), we can see that \bar{x} strictly decreases in β and converges to x^* as $\beta \rightarrow 1$ given any $\alpha > 0$. In other words, the investment strategy becomes efficient

⁹One way to understand the presence of incumbents who invested 0 is to use the convergence analysis in section 4.4. Before reaching the steady state, entrants mix between investing $c(\bar{x})$ and zero, and the non-investing sellers never trade.

in the limit. Moreover, trade becomes efficient in the limit, as the buyer's trading probability per-period (measured by $F(\bar{x})$) is strictly positive in active and partial-active equilibrium. As a result, the social welfare converges to the first best in the limit.

Lemma 6. *In the steady state equilibrium with two-sided offers and observable investments, as β converges to one, $F_e(x)$ converges in distribution to a point mass at x^* . In addition, the social welfare converges to the first best outcome.*

As we will see in the next section, the social welfare obtained in equilibrium when investments are unobservable also converge to the first best outcome. To avoid repetition, we will postpone the discussion on convergence to later sections.

6.2 Steady State Equilibrium

Let us now turn to the steady state equilibrium with unobservable investments. For any x on the support of $F_e(x)$, trade always takes place if it is the seller to make the offer, as the gains from trade $y_0 - x - \beta\pi - \beta U(x)$ must be non-negative to ensure a positive search stage payoff $U(x)$.

We can follow the same logic as in the baseline model to verify that the support of $F_e(x)$ is $[x^*, \bar{x}]$ with a mass point at x^* , and that buyers play a mixed pricing strategy over $[r_S(x^*), r_S(\bar{x})]$.

Similar to the observable benchmark case, the equilibrium could take one of the two forms (we will again call them active equilibrium and partial-active equilibrium), depending on whether $\alpha(y_0 - \bar{x})$ is larger than $c(\bar{x})$. In the active equilibrium, the support of $F(x)$ is $[x^*, \bar{x}]$. The ex ante payoffs v and π must be positive and satisfy the following two equations:

$$U(\bar{x}) = \frac{\alpha(y_0 - \bar{x} - \beta\pi)}{1 + \alpha\beta - \beta} = c(\bar{x}) + v$$

$$\frac{1 - \alpha\beta}{1 - \alpha}\pi = y_0 - \bar{x} - \beta c(\bar{x}) - \beta v$$

Solve v and π

$$v = \alpha(y_0 - \bar{x}) - c(\bar{x})$$

$$\pi = (1 - \alpha)(y_0 - \bar{x})$$

In the active equilibrium, v being weakly positive is equivalent to $\alpha(y_0 - \bar{x}) - c(\bar{x}) \geq 0$. When $\alpha(y_0 - \bar{x}) < c(\bar{x})$, the equilibrium is the partial-active equilibrium and some of incumbent sellers have invested 0 and never trade. We can verify that $F(x_0)$ and equilibrium payoffs coincide with the ones in the observable benchmark.

6.3 Comparative Statics and Convergence

As the investment strategy has support $[x^*, \bar{x}]$ and the social welfare coincide with the observable benchmark, the following proposition holds.

Proposition 7. *In the unique steady state equilibrium with two-sided offers, \bar{x} strictly decreases in β and α . Given any $\alpha > 0$, the investment strategy converges in distribution to a mass point at x^* and the equilibrium social welfare converges to the first best outcome as $\beta \rightarrow 1$.*

Notice that a seller is the residual claimant of his or her investments when making the offer (with probability α). As the time between two meetings shrinks to zero, a positive α (can be small) means that the seller has the chance to make the offer almost immediately after entry. Therefore he or she becomes the full residual claimant and invests efficiently, although he or she obtains only an α share of the total gains from trade. Combined with the fact that trade also becomes efficient in the limit, the social welfare converges to the first best.

The rate of convergence, however, depends on how likely sellers make offer. Define $\hat{\alpha} = \frac{c(x^*)}{y_0 - x^*}$. The social welfare converges at a faster rate when $\alpha \geq \hat{\alpha}$. It is because for a large enough β , we have active equilibrium when $\alpha > \hat{\alpha}$ and partial-active equilibrium when $\alpha < \hat{\alpha}$. In the latter case, some incumbent sellers who have invested zero congest the market and cause delay in trade. The rate of convergence, as a result, is slower when α is small.

Proposition 8. *With $\alpha \geq \hat{\alpha}$, the social welfare converges to $y_0 - x^* - c(x^*)$ at a rate of β^2 . With $\alpha < \hat{\alpha}$, the social welfare converges to $y_0 - x^* - c(x^*)$ at a rate of β .*

One can notice that the limit of social welfare is $y_0 - x_0$ when $\alpha = 0$ and it is $y_0 - x^* - c(x^*)$ when $\alpha > 0$. Because the equilibrium is continuous in both α and β , this suggests that the social welfare increases arbitrarily fast in α when α is close to 0 and β is close to 1. The following corollary confirms this conjecture. Intuitively, this stems from the fact that a seller becomes a residual claimant with any $\alpha > 0$ as search frictions vanish.

Corollary 2. $\frac{\partial(v+\pi)}{\partial\alpha} \rightarrow +\infty$ at $\alpha = 0$ when $\beta \rightarrow 1$.

7 Other Extensions and Robustness

7.1 Other Extensions

Investments are Observable with a Positive Probability. As the previous analysis shows, the social welfare that could be generated from positive investments is completely dissipated by delays in trade. If in the search stage investments become observable with a probability $q > 0$ per period, then profitable trades take place with no delay when that happens. We therefore expect that the social welfare under partial information is strictly higher than that under no information (or full information).

As investments are partially unobservable, seller entrants continue to use a mixed investment strategy. Its support is now $[\underline{x}, x_0]$, with $\underline{x} > x^*$ uniquely pinned down by

$$c'(\underline{x}) = \frac{-(1-q)}{1-q\beta}$$

That is, the most efficient seller now underinvests. It is because when their investments become observable, they are fully extracted. It is also easy to see that the seller's ex ante payoff equals zero, as x_0 is still on the support. The buyer's

payoff, on the other hand, is now strictly higher than that in the baseline model.

$$\pi = (y_0 - x_0)(1 - q) + q \int_{\underline{x}}^{x_0} [y_0 - x - \beta c(x)] dF(x) > y_0 - x_0$$

As a result, the social welfare is strictly higher than that in the baseline model (and in the perfectly observable benchmark). This confirms the earlier conjecture.

Buyers Costly Verify Sellers' Type. Sometimes investments become observable as a result of costly verification. To explore this possibility, we assume that before making any offer in a meeting, the buyer can pay a verification cost $A > 0$ to perfectly observe the seller's production cost. To make it non-trivial, assume that A is not too large so that a buyer is willing to verify in some situations. Now the buyer's strategy also includes the verification decision. Let us use $a \in [0, 1]$ to denote the probability of verification.

First, the net surplus from trade $y_0 - x - \beta U(x) - \beta \pi$ must be positive for any x on the support. Therefore, after paying the verification cost, a buyer offers the reserve price $x + \beta U(x)$. This is exactly why a buyer may be willing to pay the verification cost. Instead of paying a high price x_0 (or a lower price with a smaller trading probability), he or she can trade at a lower price. In addition, the probability of verification, a , must be strictly less than one. Otherwise, none of the seller entrants invests, which makes it optimal not to verify.

Following the same argument, the investment strategy and pricing strategy without verification must be mixed strategies. In each period, a seller receives his or her reserve price with a probability of a and receives a random price offer with a probability of $1 - a$. The type of the most efficient sellers on the support \underline{x} is therefore determined by

$$c'(\underline{x}) = \frac{-(1 - a)}{1 - a\beta}$$

For a positive a , \underline{x} is higher than x^* and is strictly increasing in a .

If a buyer decides not to verify, he or she must be indifferent between reserve prices. Like before, the indifference condition pins down the stationary cost distribution $F(x)$ with support $[\underline{x}, x_0]$, where \underline{x} is to be determined. If a buyer decides to

verify, then his or her expected payoff is

$$\pi = y_0 - (\underline{x} + \beta c(\underline{x}))F(\underline{x}) - \int_{\underline{x}}^{x_0} (x + \beta c(x))dF(x) - A$$

Therefore, if the buyer is indifferent between verifying and not verifying, it must be true that

$$x_0 = (\underline{x} + \beta c(\underline{x}))F(\underline{x}) + \int_{\underline{x}}^{x_0} (x + \beta c(x))dF(x) + A \quad (16)$$

The right hand side of (16) strictly increases in \underline{x} and is strictly larger than the left-hand side when $\underline{x} = x_0$. Therefore, (16) either 1) uniquely determines \underline{x} or 2) shows that the buyer strictly prefers not to verify if the equation has no solution. In the first case, the mixed strategy parameter a can be uniquely solved from $c'(\underline{x})$. In the second case, the equilibrium outcomes are the same as that in the baseline model.

Then most of the equilibrium properties are preserved. For instance, the agents' ex ante payoffs and social welfare are the same as before. Moreover, we can show that for any cost A , buyers choose not to verify with a large enough discount factor and hence we return to the baseline model. Intuitively, we know that the stationary cost distribution becomes less "uncertain" as the search frictions vanish. Hence, the value of information is smaller.

Lemma 7. *In the steady state equilibrium with costly verification, the seller's ex ante payoff is zero, the buyer's ex ante payoff and social welfare is $y_0 - x_0$.*

Moreover, given any $A > 0$, there exists a $\hat{\beta} \in [0, 1)$ such that for any $\beta > \hat{\beta}$, buyers choose not to verify and the equilibrium outcomes are the same as that in the baseline model.

7.2 Robustness

The intuition for mixed strategy does not rely too much on the specification of the model. Moreover, as long as the agents continue to play mixed strategy, most of the results continue to hold. In this section, we check the robustness of the results of

the baseline model against some alternative assumptions.

General One-to-One Matching Technology. The baseline model assumes that each player is definitely paired with one player from the other side in each period. The main results are robust if instead we have a general one-to-one matching technology so that the probability of not being paired in one period is positive.

To be more precise, the equilibrium investment strategy $F_e(x)$ remains non-degenerate with convex support $[\underline{x}, x_0]$ and a buyer plays a mixed pricing strategy over the reserve prices of these types. The difference is that \underline{x} is higher than x^* when a seller cannot be paired for sure in each period. Moreover, because the indifference conditions continue to hold, the equilibrium payoffs and social welfare equal the values generated with observable investments. Finally, as β converges to one, it is as if the most efficient sellers can trade immediately and hence they invest efficiently. The convergence results of $F(x)$ and $F_e(x)$ can also be extended with different rates of convergence that depend on the matching technology.

Ex ante Heterogeneous Sellers. We assume that all of the sellers are ex ante identical in the baseline model. The intuition can be extended to settings with ex ante heterogeneous sellers provided that their ex ante types remain unobservable and that their ex ante type distribution satisfies some mild assumptions.

As a simple example, suppose some fraction of the seller entrants are born with the production cost x^* and hence there is no need for them to invest. Others are born with the production cost x_0 and have the investment opportunity as specified in the baseline model. We can easily verify that as long as the fraction of the efficient type is smaller than $F_e(x^*)$ (which is easier to be satisfied when β is large enough), all of the conclusions in the baseline model continue to hold.

Exogenous Death Shock. Suppose each player experiences an exogenous death shock with positive probability. For most of the analysis, this is equivalent to redefining a smaller discount factor that also converges to one in the limit. The only complication is that now sellers also exit due to the death shock. Thus, we need to rewrite the condition of stationary distribution accordingly. In other words, the investment strategy is slightly changed.

If we denote the rate of the death shock as \tilde{r} , then the limit of $F_e(x^*)$ is continuous and decreasing in \tilde{r} , and is strictly positive for any \tilde{r} .

Lemma 8. *As the time between two consecutive meetings shrinks to zero,*

$$F_e(x^*) \rightarrow \left[1 + \frac{x_0 - x^* - c(x^*)}{y_0 - x_0} \frac{\tilde{r}}{r + \tilde{r}}\right]^{-1}$$

Therefore, the limit of $F_e(x^)$ decreases in the rate of death shock \tilde{r} and is larger than $\left[1 + \frac{x_0 - x^* - c(x^*)}{y_0 - x_0}\right]^{-1}$.*

8 Conclusion

This paper examines the investment incentive and its welfare consequences in an infinite horizon random search and bargaining game with unobservable and selfish investments.

We demonstrate that in a unique steady state equilibrium, both the investment strategy and the price offer distribution are non-degenerate with convex supports. Unobservability generates rent for high investments and therefore investment incentives exist even if the sellers have no bargaining power.

However, positive investments above the minimum level fail to generate any social welfare for any search friction. Trading inefficiency caused by unobservability erodes the welfare gain that could be created.

Moreover, we show that if the buyers have all of the bargaining power, then as meetings become more frequent the investment distributions of the incumbents and entrants shift in the opposite direction. The incumbent investment distribution converges to a point mass at no investment, and an entrant's investment becomes efficient.

Appendices

A Proofs for the Baseline Model

Appendix A.A Proof of Lemma 1

1. $U(x)$ is **Strictly Decreasing**. As a type x seller can always adopt the reserve price of a type $x + \epsilon$ seller (for some $\epsilon > 0$), $U(x)$ must be strictly decreasing in x .
2. $r_S(x)$ is **Strictly Increasing**. Multiply both sides of the seller's value function by β and add x , we obtain the following equation after rearranging:

$$\begin{aligned} r_S(x) - \beta[E(p \mid p \geq r_S(x))(1 - H(r_S(x)) + Pr(p = r_S(x))) \\ + r_S(x)(H(r_S(x)) - Pr(p = r_S(x)))] = (1 - \beta)x \end{aligned} \quad (17)$$

The left-hand side strictly increases in $r_S(x)$ and the right-hand side strictly increases in x . Therefore, $r_S(x)$ is strictly increasing in x .

3. $U(\bar{x}) = 0$. Because $r(\bar{x})$ is the highest price that a buyer is willing to offer, equation (1) with $x = \bar{x}$ becomes $U(\bar{x}) = \beta U(\bar{x})$. Hence, $U(\bar{x}) = 0$.
4. $U(x)$ and $r_S(x)$ are **Continuous**. No buyer offers a price higher than $r_S(\bar{x})$. Therefore, for any $x > \bar{x}$, $H(r_S(x)) = 1$ and $U(x) = 0$.

The previous step shows that $U(\bar{x}) = 0$.

Because $U(x)$ decreases in x for any $x < \bar{x}$, $U(x)$ can only have downward jumps. Suppose $U(x)$ jumps down at some point \hat{x} . Given that $r_S(x) = x + \beta U(x)$, $r_S(x)$ also jumps downward at \hat{x} . However, this contradicts $r_S(x)$ being strictly increasing.

Therefore, both $U(x)$ and $r_S(x)$ are continuous.

Appendix A.B Proof of Proposition 1

1. Supports of $F(x)$, $F_e(x)$ and $H(p)$.

The supports are closed given the assumption that the sellers and buyers trade when indifferent.

Next, we show that the supports are convex. First, if a price offer p is on the support of $H(p)$, then $\hat{x}(p)$ must be on the support of $F(x)$ and $F_e(x)$. Otherwise, the buyer who offers price p is not optimizing because he or she can lower the price to $r_S(x')$ without affecting the probability of trade, where x' is the highest production cost on the support that is lower than $\hat{x}(p)$.

Now suppose there exist p_1, p_2 on the support of $H(p)$, such that any $p \in (p_1, p_2)$ is not on the support. As p_1 and p_2 are on the support, $x_i = \hat{x}(p_i)$, $i = 1, 2$ must be on the support as well. For any $x \in (x_1, x_2)$, $U'(x)$ is a constant because

$$U'(x) = \frac{-1 + H(p_2)}{1 - \beta H(p_2)}$$

On the other hand, $c'(x)$ strictly increases in x . Together with the continuity of $U(x)$, it is impossible to have the indifference condition $U(x_1) - c(x_1) = U(x_2) - c(x_2) \geq U(x) - c(x)$ satisfied, for any $x \in (x_1, x_2)$. Therefore, the support of $H(p)$ is convex. By the continuity of $r_S(x)$, the support of $F(x)$ and $F_e(x)$ is also convex.

The lower bound of the price offers in equilibrium is never lower than the reserve price of the most efficient seller, i.e., $H(r_S(\underline{x})) - Pr(\tilde{p} = r_S(\underline{x})) = 0$. Plugging it into the envelope condition, $U'(\underline{x}) = c'(\underline{x}) = -1$. This implies that $\underline{x} = x^*$.

Hence, the support of $F(x)$ and $F_e(x)$ is $[x^*, x_0]$ and the support of $H(p)$ is $[r_S(x^*), r_S(x_0)]$.

2. $H(p)$ has no atom.

$U(x)$ is differentiable for any x on the support, because 1) the support is convex, 2) $U(x) - c(x) = 0$ and 3) $c(x)$ is differentiable. This implies that

$r_S(x)$ and $\hat{x}(p)$ are differentiable for any x and p on the support. Hence, we can solve $H(p)$ from the equilibrium condition $U'(x) = c'(x)$ as follows:

$$H(r_S(x)) = \frac{1 + c'(x)}{1 + \beta c'(x)} \Rightarrow H(p) = \frac{1 + c'(\hat{x}(p))}{1 + \beta c'(\hat{x}(p))} \quad (18)$$

It is straightforward to verify that $H(p)$ has no atom.

3. $F(x)$ and $F_e(x)$ have a unique point mass at x^* .

A buyer who offers $r_S(x^*)$, which is only accepted by sellers with a cost of x^* , would get zero payoff if $F(x^*) = 0$. If this is the case, then the buyer would find it profitable to deviate to offering $r_S(x_0) = x_0$ which yields a positive payoff. Therefore, there must be a point mass at x^* .

Next we show that x^* is the unique point mass. Suppose there is another point mass at $x \in (x^*, x_0]$. Then there exist an ϵ , such that buyers would rather not offer price $p \in (r_S(x - \epsilon), r_S(x))$. By raising the price by a small amount, buyers can enjoy a discontinuous upward jump of the trading probability. This contradicts the convexity property of the support.

Appendix A.C Proof of Proposition 3

$F(x)$ has first order stochastic dominance over $F_e(x)$ if $F_e(x) - F(x) \geq 0$ for all x , with strict inequality for some x .

$$\begin{aligned} F_e(x) - F(x) &= \frac{F(x) - \int_{x^*}^x H(r_S(\tilde{x}))dF(\tilde{x})}{1 - \int_{x^*}^{x_0} H(r_S(\tilde{x}))dF(\tilde{x})} - F(x) \\ &= F(x) \frac{\int_{x^*}^{x_0} H(r_S(\tilde{x}))dF(\tilde{x}) - \int_{x^*}^x H(r_S(\tilde{x}))d\frac{F(\tilde{x})}{F(x)}}{1 - \int_{x^*}^{x_0} H(r_S(\tilde{x}))dF(\tilde{x})} \end{aligned}$$

Therefore, $F_e(x) - F(x) \geq 0$ for any $x \in [x^*, x_0]$ and the inequality is strict except for $x = x_0$.

Appendix A.D Proof of Theorem 2

1. We have shown the first two parts of the theorem.
2. By (5), the proportion of seller entrants who invest efficiently is

$$\begin{aligned}
 F_e(x^*) &= \frac{F(x^*)}{1 - \int_{x^*}^{x_0} H(r_S(x))f(x)dx} \\
 &= [1 + \frac{1 - F(x^*)}{F(x^*)} \int_{x^*}^{x_0} (1 - H(r_S(x))) \frac{f(x)}{1 - F(x^*)} dx]^{-1} \\
 &= [1 + A]^{-1}
 \end{aligned}$$

where $A = \frac{1 - F(x^*)}{F(x^*)} \int_{x^*}^{x_0} (1 - H(r_S(x))) \frac{f(x)}{1 - F(x^*)} dx$.

By Median Value Theorem, there exist a $\tilde{x} \in (x^*, x_0)$ such that

$$\begin{aligned}
 A &= \frac{1 - F(x^*)}{F(x^*)} [1 - H(r_S(\tilde{x}))] \\
 &= \frac{-c'(\tilde{x})(x_0 - x^* - \beta c(x^*))}{(y_0 - x_0)(1 + \beta c'(\tilde{x}))}
 \end{aligned}$$

Take derivative with respect to β

$$\frac{\partial A}{\partial \beta} = \frac{-c''(\tilde{x}) \frac{\partial \tilde{x}}{\partial \beta} (x_0 - x^* - \beta c(x^*)) + c'(\tilde{x}) [c(x^*) + c'(\tilde{x})(x_0 - x^*)]}{(y_0 - x_0)(1 + \beta c'(\tilde{x}))^2}$$

Therefore, the sufficient conditions for the derivative to be negative are

$$\frac{\partial \tilde{x}}{\partial \beta} > 0 \text{ and } c(x^*) + c'(\tilde{x})(x_0 - x^*) > 0$$

We first show that the first condition is satisfied. \tilde{x} is implicitly defined by

$$\int_{x^*}^{x_0} \frac{1 - H(r_S(x))}{1 - \beta} d \frac{F(x)}{1 - F(x^*)} = \frac{-c'(\tilde{x})}{1 + \beta c'(\tilde{x})} \quad (19)$$

As the preceding analysis shows, $\frac{1 - H(r_S(x))}{1 - \beta}$ strictly decreases in β for any $x \in (x^*, x_0)$. In addition, the mass of the conditional distribution $F(x | x >$

x^*) shifts to higher x 's, where $\frac{1-H(r_S(x))}{1-\beta}$ is smaller. As a result, the left-hand side of (19) strictly decreases in β . At the same time,

$$\frac{\partial \frac{-c'(\tilde{x})}{1+\beta c'(\tilde{x})}}{\partial \beta} = \frac{-c''(\tilde{x}) \frac{\partial \tilde{x}}{\partial \beta} + [c'(\tilde{x})]^2}{(1 + \beta c'(\tilde{x}))^2}$$

Since the derivative has to be negative so that (19) holds, $\frac{\partial \tilde{x}}{\partial \beta}$ must be strictly positive.

This in turn implies that we only need to verify the second sufficient condition when $\beta = 0$, as $c'(\tilde{x})$ is the smallest when $\beta = 0$. It is straight forward to check that the density of the stationary distribution strictly increases in x , as

$$f(x) = \frac{(y_0 - x_0 - \beta\pi)(1 + \beta c'(x))}{[y_0 - \beta\pi - x - \beta c(x)]^2}$$

When $\beta = 0$, $1 - H(r_S(x)) = -c'(x)$. Therefore, equation (19) becomes,

$$\begin{aligned} -c'(\tilde{x}) &= \int_{x^*}^{x_0} -c'(x) \frac{f(x)}{1 - F(x^*)} dx \\ \text{where the right-hand-side} &< \int_{x^*}^{x_0} -c'(x) \frac{1}{x_0 - x^*} dx = \frac{c(x^*)}{x_0 - x^*} \\ &\Rightarrow c(x^*) + c'(\tilde{x})(x_0 - x^*) > 0 \end{aligned}$$

We have proved that both of the sufficient conditions hold and thus

$$\frac{\partial F_e(x^*)}{\partial \beta} > 0$$

Next, we prove that $F_e(x^*) \rightarrow 1$ in the limit. \tilde{x} is defined by

$$\int_{x^*}^{x_0} (1 - H(r_S(x))) \frac{f(x)}{1 - F(x^*)} dx = [1 - H(r_S(\tilde{x}))] \quad (20)$$

For any $\epsilon \in (0, x_0 - x^*)$, we can rewrite the left-hand side of (20) as

$$\begin{aligned} & \int_{x^*}^{x_0 - \epsilon} (1 - H(r_S(x))) \frac{f(x)}{1 - F(x^*)} dx + \int_{x_0 - \epsilon}^{x_0} (1 - H(r_S(x))) \frac{f(x)}{1 - F(x^*)} dx \\ &= [1 - H(r_S(x_1))] \frac{F(x_0 - \epsilon) - F(x^*)}{1 - F(x^*)} + [1 - H(r_S(x_2))] \frac{1 - F(x_0 - \epsilon)}{1 - F(x^*)} \end{aligned} \quad (21)$$

where $x_1 \in (x^*, x_0 - \epsilon)$ and $x_2 \in (x_0 - \epsilon, x_0)$.

Here $F(x_0 - \epsilon) - F(x^*)$ can be rewritten as,

$$(1 - \beta)(y_0 - x_0) \frac{-x^* - \beta c(x^*) + (x_0 - \epsilon) + \beta c(x_0 - \epsilon)}{(y_0 - \beta\pi - x_0 + \epsilon - \beta c(x_0 - \epsilon))(y_0 - \beta\pi - x^* - \beta c(x^*))}$$

We can verify that for any ϵ and ϕ , there exists an $\eta > 0$, such that when $1 - \beta < \eta$, $F(x_0 - \epsilon) - F(x^*) < \phi$. Given that $1 - H(r_S(x))$ is bounded for any x , \tilde{x} converges to x_2 in the limit. Because x_2 is in the interval $(x_0 - \epsilon, x_0)$, for small enough ϵ , \tilde{x} converges to x_0 .

Therefore, A converges to zero in the limit. This in turn implies that $F'_e(x^*) = [1 + A]^{-1}$ converges to one.

Appendix A.E Proof of Proposition 4

In the first period, the measure of exits is e_1 , which is also the proportion of incumbents who exit the market in the following periods. Denote the measure of incumbents at the beginning of period t by u_t . Then for any $t \geq 1$

$$u_{t+1} = 1 + u_t(1 - e_t)$$

Multiply both sides by e_1 and rearrange, the above equation becomes

$$e_{t+1} = e_t + e_1(1 - e_t)$$

First, we can verify that $e_t < 1$ for any t . When $t = 1$, e_1 is strictly less than 1 by construction. For any $t > 1$, the above condition shows that e_t is a convex

combination of 1 and e_1 and hence is also strictly less than 1.

This equation also implies that $\{e_t\}$ is an increasing sequence, as $e_{t+1} - e_t = e_1(1 - e_t) > 0$.

An increasing and bounded sequence must have a limit. Denote the limit as e_∞ , which can be solved from

$$e_\infty = e_\infty + e_1(1 - e_\infty) \Rightarrow e_\infty = 1.$$

The rest of the proposition is already proved in the main body of this paper.

B Proofs for the Two-Sided Investment Case

Appendix B.A Proof of Lemma 2

1. $p(y)$ is increasing.

If the support of $G(y)$ is degenerate, then we have nothing to prove.

Otherwise, consider any y_1 and y_2 on the support with $y_1 > y_2$. Denote p_i as $p(y_i)$ for $i = 1, 2$. Then the following two inequalities must hold

$$\begin{aligned} (y_1 - p_1)F(\hat{x}(p_1)) + [1 - F(\hat{x}(p_1))]\beta\Pi(y_1) &\geq (y_1 - p_2)F(\hat{x}(p_2)) + [1 - F(\hat{x}(p_2))]\beta\Pi(y_1) \\ (y_2 - p_2)F(\hat{x}(p_2)) + [1 - F(\hat{x}(p_2))]\beta\Pi(y_2) &\geq (y_2 - p_1)F(\hat{x}(p_1)) + [1 - F(\hat{x}(p_1))]\beta\Pi(y_2) \end{aligned}$$

Adding the two inequalities

$$(y_1 - y_2)[F(\hat{x}(p_1)) - F(\hat{x}(p_2))] \geq [F(\hat{x}(p_1)) - F(\hat{x}(p_2))][\beta\Pi(y_1) - \beta\Pi(y_2)]$$

In equilibrium, $\Pi(y_1) - \Pi(y_2) = e(y_1) - e(y_2)$, which implies that $y_1 - y_2 > \beta[\Pi(y_1) - \Pi(y_2)]$. Given this, the above inequality then shows that $F(\hat{x}(p_1)) \geq F(\hat{x}(p_2))$. This proves that $p(y)$ increases in y .

2. $P(y)$ is single-valued.

We can easily prove that $F(x)$ is a strictly increasing function of x for any $x \in [x^*, x_0]$. Then consider any y on the support of $G_e(y)$ and denote the corresponding optimal price as p . The search stage payoff is

$$\Pi(y) = \max_p \frac{[y - p]F(\hat{x}(p))}{1 - \beta[1 - F(\hat{x}(p))]}$$

which is strictly convex in p . Therefore, $P(y)$ is single-valued.

Appendix B.B Proof of Proposition 5

1. We can apply the same proof in the baseline model to show that the support of $F_e(x)$ and $F(x)$ is $[x^*, x_0]$ and that x^* is the unique point mass.
2. The compactness of the support of $G_e(y)$ and $G(y)$ results from the fact that y^* is finite and that all of the agents choose to trade when feeling indifferent.

To see the convexity property, suppose that y_1, y_2 are on the support and any $y \in (y_1, y_2)$ is not. Then it must be the case that $p(y_1) < p(y_2)$. Otherwise $\Pi'(y)$ is constant in the interval and the indifference condition cannot be satisfied. Combined with the monotonicity of $p(y)$, the fact that $y \in (y_1, y_2)$ are on the support implies that $p \in (p(y_1), p(y_2))$ are not on the support of $H(p)$. This leads to a contradiction.

Because the support of $G(y)$ is convex, we can use the envelope and indifference condition to determine \bar{y} and \underline{y} .

$$e'(\bar{y}) = \Pi'(\bar{y}) = 1 \text{ and } e'(\underline{y}) = \Pi'(\underline{y}) = \frac{F(x^*)}{1 - \beta(1 - F(x^*))}$$

As a result, $\bar{y} = y^*$ and $\pi = y^* - x_0 - e(y^*)$. Plug π into the condition for \underline{y} , it becomes

$$y^* - x_0 - e(y^*) = (\underline{y} - x^* - \beta c(x^*))e'(\underline{y}) - e(\underline{y})$$

\underline{y} is uniquely determined by the above equation and $\underline{y} < y^*$. To see this,

notice that the right-hand side of the above equation strictly increases in \underline{y} . It equals zero when $\underline{y} = y_0$ and it is strictly larger than the left-hand side when $\underline{y} = y^*$. Therefore, two sides of the equation can cross each other only once and the intersection point is strictly smaller than y^* .

Finally, no point mass on the support of $H(p)$ implies that the same property holds for $G(y)$.

Appendix B.C Proof of Lemma 3

Combining the envelope condition for $\Pi(y)$ and the indifference condition that $\Pi'(y) = e'(y)$ for any y , we have

$$F(\hat{x}(p(y))) = \frac{(1 - \beta)e'(y)}{1 - \beta e'(y)} \text{ for any } y \text{ on the support}$$

We can use the indifference condition $\Pi(y) - e(y) = \pi$ to solve $p(y)$ as follows:

$$\begin{aligned} \Pi(y) - e(y) &= (y - p(y))e'(y) - e(y) = y^* - x_0 - e(y^*) \\ \Rightarrow p(y) &= y - \frac{e(y) + y^* - x_0 - e(y^*)}{e'(y)} \end{aligned}$$

$p(y)$ is continuous and strictly increasing in y . Therefore, the inverse function $y(p)$ is well defined. We can then plug it in to solve $F(x)$ and $G(y)$.

From the preceding discussion, we can see that $F(x)$, $F_e(x)$, $G(y)$ and $G_e(y)$ as defined are the only distributions that satisfy all equilibrium conditions. Hence, the steady state equilibrium is unique.

Appendix B.D Proof of Proposition 6

1. By (10), the implicit function theorem tells us that

$$[\underline{y} - x^* - \beta c(x^*)]e''(\underline{y})\frac{\partial \underline{y}}{\partial \beta} = c(x^*)e'(\underline{y})$$

Therefore, $\frac{\partial \underline{y}}{\partial \beta}$ is strictly positive.

Denote the $t * 100^{th}$ percentile of $G(y)$ with β as $y_{t,\beta}$, i.e.,

$$G(y_{t,\beta}) = \frac{1 + c'(\hat{x}(p(y_{t,\beta})))}{1 + \beta c'(\hat{x}(p(y_{t,\beta})))} = t$$

Rearrange the above equation

$$1 - t = -(1 - t\beta)c'(\hat{x}(p(y_{t,\beta})))$$

Therefore, when β increases, $\hat{x}(p(y_{t,\beta}))$ strictly decreases.

2. $\lim_{\beta \rightarrow 1} \underline{y} < y^*$ can be shown by plugging $\beta = 1$ and $\underline{y} = y^*$ into equation (10), and check that the left-hand side is strictly smaller the right-hand side.

For any $y > \underline{y}$, $\hat{x}(p(y)) > x^*$. Therefore, the denominator of $G(y)$, $1 + c'(\hat{x}(p(y)))$, is strictly positive. Then it is straightforward to verify that as $\beta \rightarrow 1$, $G(y) \rightarrow 1$ for any $y > \underline{y}$.

For any $y > \underline{y}$, there exist \check{y} and \tilde{y} , such that

$$\begin{aligned} G_e(y) &= \frac{\int_{\underline{y}}^y F(\hat{x}(p(\tilde{y})))dG(\tilde{y})}{\int_{\underline{y}}^{y^*} F(\hat{x}(p(\tilde{y})))dG(\tilde{y})} = \frac{F(\hat{x}(p(\tilde{y})))G(y)}{F(\hat{x}(p(\tilde{y})))} \\ &= \frac{e'(\tilde{y})(1 - \beta e'(\tilde{y}))}{e'(\tilde{y})(1 - \beta e'(\tilde{y}))}G(y) \end{aligned}$$

When $\beta \rightarrow 1$, both \check{y} and \tilde{y} approaches \underline{y} , following the same argument in the proof for proposition 2, and $G(y)$ approaches one for any $y > \underline{y}$. Therefore, $G_e(y) \rightarrow 1$ for any $y > \underline{y}$.

C Proofs for the Two-Sided Offer Case

Appendix C.A Proof of Lemma 4

Suppose there are more than one optimal investment levels in a stationary equilibrium. Given condition (15), there can be at most one x with non-negative net

surplus.

This means that if there exists another production cost of \hat{x} on the support, the associated surplus from trade must be negative. As a result, a seller of cost \hat{x} can never trade and his or her ex ante payoff is $v = -c(\hat{x})$. So the only possibility is that $\hat{x} = x_0$.

The sellers with the production cost x_0 never leave the market. Therefore, the entrant's type can only be the invested one in a stationary equilibrium.

Appendix C.B Proof of Lemma 5

Equilibrium 1.

Assume $\alpha(y_0 - \bar{x}) \geq c(\bar{x})$.

All seller entrants invest to decrease their production costs to \bar{x} . Agents' ex ante payoffs can be solved as $v = \alpha(y_0 - \bar{x}) - c(\bar{x})$ and $\pi = (1 - \alpha)(y_0 - \bar{x})$. By the assumption, v is positive and all seller entrants invest positive amount.

Equilibrium 2.

Assume $\alpha(y_0 - \bar{x}) \leq c(\bar{x})$.

If a seller entrant invests, he or she chooses to decrease the production cost to \bar{x} . Given π , his or her ex ante payoff $v = \frac{\alpha}{1 + \alpha\beta - \beta}(y_0 - \bar{x} - \beta\pi) - c(\bar{x})$. To make sure that sellers are indifferent between \bar{x} and x_0 , v must equal zero. As a consequence,

$$\pi = \frac{\alpha(y_0 - \bar{x}) - (1 + \alpha\beta - \beta)c(\bar{x})}{\alpha\beta}$$

Denote $F(\bar{x})$ as q , we can also solve π from the buyer's value function as follows:

$$\pi = \alpha\beta\pi + (1 - \alpha)[q(y_0 - \bar{x} - \beta c(\bar{x})) + (1 - q)\beta\pi]$$

Equating two π 's, q can be solved as:

$$q = \frac{(1 - \beta)\pi}{(1 - \alpha)(y_0 - \bar{x} - \beta c(\bar{x})) - \beta\pi}$$

As q is a probability, it lies between zero and one. q is always positive by the

above equation. The requirement that q is less than one is equivalent to $\alpha(y_0 - \bar{x}) \leq c(\bar{x})$.

The last equilibrium condition we need to verify is $y_0 - x_0 - \beta\pi \leq 0$. After plugging in the expression of π , this condition becomes $c(\bar{x}) \leq \frac{\alpha}{1+\alpha\beta-\beta}(x_0 - \bar{x})$. This inequality holds since $c'(\bar{x}) = \frac{-\alpha}{1+\alpha\beta-\beta}$ and $c(x)$ is strictly convex.

Appendix C.C Proof of Lemma 6

As $\beta \rightarrow 1$, because α is bounded above zero, the right-hand side of (15) converges to -1 , which implies $\bar{x} \rightarrow x^*$. Because $F_e(x)$ is a point mass at \bar{x} in both types of equilibrium, it converges to a point mass at x^* .

In equilibrium 1, the social welfare equals $y_0 - \bar{x} - c(\bar{x})$. Hence, it converges to the first best. In equilibrium 2, the social welfare equals π , which also converges to $y_0 - x^* - c(x^*)$, given that α is bounded above zero.

Appendix C.D Proof of Proposition 8

First we show that active equilibrium is the unique equilibrium for any $\alpha \geq \hat{\alpha}$. By the definition of $\hat{\alpha}$, $\alpha(y_0 - x^*) - c(x^*)$, which is the limit of $\alpha(y_0 - \bar{x}) - c(\bar{x})$, must be strictly positive. In addition, it is easy to see that $\alpha(y_0 - \bar{x}) - c(\bar{x})$ strictly decreases in β . Therefore, $\alpha(y_0 - \bar{x}) - c(\bar{x})$ is strictly positive for any $\beta \in [0, 1)$ and the equilibrium is the active equilibrium.

Denote the social welfare as $s(\beta)$ and its limit as $s(1)$.

$$\lim_{\beta \rightarrow 1} \frac{|s(\beta) - s(0)|}{1 - \beta^2} = \frac{(1 - \alpha)^2}{2\alpha^2 c''(x^*)} \in (0, \infty)$$

This proves that the social welfare converges at a rate of β^2 for $\alpha \geq \hat{\alpha}$.

Next, it is easy to see that there exist a $\hat{\beta}(\alpha) \in [0, 1)$ for any $\alpha < \hat{\alpha}$, such that partial-active equilibrium is the unique equilibrium for any $\beta \in (\hat{\beta}(\alpha), 1)$. This comes from the definition of $\hat{\alpha}$ and the fact that $\alpha(y_0 - \bar{x}) - c(\bar{x})$ strictly decreases in β .

Then when β is close enough to 1,

$$s(\beta) = \frac{1}{\beta}(y_0 - \bar{x}) - \frac{1 + \alpha\beta - \beta}{\alpha\beta}c(\bar{x})$$

Hence

$$\lim_{\beta \rightarrow 1} \frac{|s(\beta) - s(0)|}{1 - \beta} = -[y_0 - x^* - \frac{1}{\alpha}c(x^*)] \in (0, \infty)$$

This proves that the social welfare converges at a rate of β for $\alpha < \hat{\alpha}$.

Appendix C.E Proof of Corollary 2

We have already shown that when $\alpha < \hat{\alpha}$ and β is close enough to 1, the equilibrium is the partial-active equilibrium. Then take derivative of the social welfare s with respect to α ,

$$\frac{\partial s}{\partial \alpha} = \frac{1 - \beta}{\alpha^2 \beta} c(\bar{x})$$

Evaluated at $\alpha = 0$, the derivative goes to infinity.

D Proofs for Robustness and Other Extensions

Appendix D.A Proof of Lemma 7

x_0 is on the support. Hence, the first part of the proposition holds.

As $\beta \rightarrow 1$, $F(x) \rightarrow 0$ for any $x < x_0$. Therefore, the right-hand side of (16) converges to $x_0 + A$, which is strictly larger than the left-hand side. This means that a buyer does not pay to verify if β is close enough to 1.

Moreover, the right-hand side shifts up with a larger β . This follows from 1) $F(x)$ increases in β , which is equivalent to putting higher weights on larger reserve prices and 2) the reserve price $x + \beta c(x)$ increases in β for any x .

Given the above two facts, there exists a $\hat{\beta}$ for any A , such that the buyers choose not to verify when $\beta > \hat{\beta}$.

Appendix D.B Proof of Lemma 8

Denote the probability of surviving a death shock as $\delta = e^{-\tilde{r}t}$, where t is the length of the time between two periods. Then

$$\begin{aligned} F_e(x^*) &= \frac{F(x^*)}{1 - \delta \int_{x^*}^{x_0} H(r_S(x)) f(x) dx} \\ &= [1 + \frac{1 - F(x^*)}{F(x^*)} \int_{x^*}^{x_0} [1 - \delta H(r_S(x))] \frac{f(x)}{1 - F(x^*)} dx]^{-1} \\ &= [1 + B]^{-1} \end{aligned}$$

Plug in $F(x^*)$ and $H(r_S(x))$, and use the Median Value Theorem, there exists a $\tilde{x} \in (x^*, x_0)$ such that

$$B = \frac{x_0 - x^* - \hat{\beta}c(x^*)}{(1 - \hat{\beta})(y_0 - x_0)} \frac{1 - \delta + (\hat{\beta} - \delta)c'(\tilde{x})}{1 + \hat{\beta}c'(\tilde{x})}$$

Here $\hat{\beta} = \beta\delta$. Take $t \rightarrow 0$,

$$\lim_{t \rightarrow 0} B = \frac{x_0 - x^* - c(x^*)}{(y_0 - x_0)} \lim_{t \rightarrow 0} \left[\frac{1}{1 + e^{-(\tilde{r}+r)t} c'(\tilde{x}_t)} + \frac{1 + c'(\tilde{x}_t)}{1 + e^{-(\tilde{r}+r)t} c'(\tilde{x}_t)} \frac{e^{-rt} - 1}{1 - e^{-(\tilde{r}+r)t}} \right]$$

By the same argument in the proof for Theorem 2, the limit of \tilde{x}_t is x_0 . Using the L'Hopital's rule

$$\lim_{t \rightarrow 0} B = \frac{x_0 - x^* - c(x^*)}{y_0 - x_0} \frac{\tilde{r}}{r + \tilde{r}}$$

References

- [1] Acemoglu, D., 1996. "A Microfoundation for Social Increasing Returns in Human Capital Accumulation," *The Quarterly Journal of Economics* 111: 779-804.
- [2] Acemoglu, D. and R. Shimer, 1999. "Holdups and Efficiency with Search Frictions," *International Economic Review* 40(4): 827-49.

- [3] Albrecht, J. and S. Vroman, 1992. "Non-Existence of Single-Wage Equilibria in Search Models with Adverse Selection", *The Review of Economic Studies* 59: 617-624.
- [4] Albrecht, J. and S. Vroman, 2002. "A Matching Model with Endogenous Skill Requirements," *International Economic Review, Department of Economics* 43(1): 283-305.
- [5] Burdett, K. and K.L. Judd, 1983. "Equilibrium Price Dispersion," *Econometrica* 51: 955-969.
- [6] Burdett, K. and D.T. Mortensen, 1998. "Wage Differentials, Employer Size, and Unemployment," *International Economic Review* 39(2): 257-73.
- [7] Davis, S., 2001. "The Quality Distribution of Jobs and the Structure of Wages in Search Equilibrium," NBER Working Papers 8434.
- [8] Che, Y. and D.B. Hausch, 1999. "Cooperative Investments and the Value of Contracting," *The American Economic Review* 89(1): 125-147.
- [9] Cole, H., G. Mailath and A. Postlewaite, 2001. "Efficient Non-Contractible Investments in Large Economies," *Journal of Economic Theory* 101: 333-373.
- [10] Diamond, P.A., 1971. "A model of price adjustment," *Journal of Economic Theory* 3: 156-168.
- [11] Grossman, S.J. and O.D. Hart, 1986. "The Costs and Benefits of Ownership: A Theory of Vertical and Lateral Integration," *Journal of Political Economy* 94: 691-719.
- [12] Grout, P., 1984. "Investment and Wages in the Absence of Binding Contracts: A Nash Bargaining Approach," *Econometrica* 52: 449-460.
- [13] Gul, F., 2001. "Unobservable Investment and the Hold-Up Problem," *Econometrica* 69: 343-376.
- [14] Hart, O. and J. Moore, 1990. "Property Rights and the Nature of the Firm," *Journal of Political Economy* 98: 1119-58.
- [15] Hermalin, B., 2013. "Unobserved Investment, Endogenous Quality, and Trade," *RAND Journal of Economics* 44: 33-55.
- [16] Hermalin, B. and M. Katz, 2009. "Information and the Hold-Up Problem," *RAND Journal of Economics*, 40: 405-423.

- [17] Lau, S., 2008. "Information and Bargaining in the Hold-Up Problem," *RAND Journal of Economics*, 39: 266-282.
- [18] Lauerermann, S., 2012. "Asymmetric Information in Bilateral Trade and in Markets: An Inversion Result," *Journal of Economic Theory* 147(5): 1969-1997.
- [19] Lauerermann, S., 2013. "Dynamic Matching and Bargaining Games: A General Approach," *American Economic Review* 103(2): 663-89.
- [20] Mailath, G., A. Postlewaite and L. Samuelson, 2013. "Pricing and Investments in Matching Markets," *Theoretical Economics* 8: 535-590.
- [21] Mortensen, D., 2005. *Wage Dispersion: Why Are Similar Workers Paid Differently?* (Cambridge: MIT Press).
- [22] Rogerson, W., 1992. "Contractual Solutions to the Hold-Up Problem," *Review of Economic Studies* 59: 777-794.
- [23] Satterthwaite, M. and A. Shneyerov, 2007. "Dynamic Matching, Two-Sided Incomplete Information, and Participation Costs: Existence and Convergence to Perfect Competition," *Econometrica* 75: 155-200.
- [24] Shneyerov, A. and A.C.L. Wong, 2010. "Bilateral matching and bargaining with private information," *Games and Economic Behavior* 68: 748-762.
- [25] Rubinstein, A. and A. Wolinsky, 1990. "Decentralized Trading, Strategic Behaviour and the Walrasian Outcome," *The Review of Economic Studies* 57: 63-78.
- [26] Williamson, O., 1975. *Markets and Hierarchies: Analysis and Antitrust Implications* (New York: Free Press).