Unobservable Investments and Search Frictions

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Abstract

This paper develops a pre-entry investment and random search framework to jointly study the investment incentives and trading efficiency. A seller entrant can make unobservable investments to decrease the production cost before searching for buyers. In the unique steady state equilibrium, investment and price dispersion emerge simultaneously with ex ante identical buyers and sellers. Despite the positive investments, when buyers make take-it-or-leave-it offers, the equilibrium payoffs and social welfare are constant given any search friction and equal to the equilibrium values when investments are observable (indicating no investment). This novel property remains true even when the investment strategy becomes socially optimal as search frictions diminish.

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1 Introduction

Investment incentives and trading outcomes with search frictions have been vastly studied in the holdup and search literature, respectively. This paper investigates an environment in which these two issues are naturally connected: sellers need to decide how much to invest to decrease their production costs and then start to search for buyers. Sellers are incentivized by the return on investments that they can reap, which is determined through search and bargaining in the market. At the same time, trading outcomes depend on investments, as that govern the characteristics of entrants. By exploring this interdependency, we link the Diamond Paradox with the insights into the interplay of investments and bargaining. We show that a lot of equilibrium predictions on investments and trade efficiency are drastically different if we overlook this interdependency.

In this paper, we focus on large markets in which a great number of buyers and sellers search for trading partners. In a large market, the investments are less likely to be relationship-specific or perfectly observable by buyers.[1] We should therefore expect the seller entrants to have stronger incentive to invest in the described environment than in an environment with relation-specific and observable investments. It is indeed the case in equilibrium. Surprisingly, we find that such positive investments do not necessarily lead to welfare improvement under realistic settings. In particular, when buyers have all of the bargaining power, there is no welfare gain at all. This result remains true even when the investments become socially optimal as search frictions vanish.

The model we propose is a discrete time infinite horizon random search model with exogenous entry and pre-entry investments. At the beginning of each period, there is one unit mass of ex ante identical sellers and buyers entering the market. A buyer entrant demands one unit of output and receives utility $y_0 > 0$ from consuming it. A seller entrant is endowed with the technology to produce one unit of

[1] An example could be a market that involves a large number of intermediate suppliers and many retailers. A supplier can invest to find cheaper producers before searching for retailers. Such investments are not relation-specific, as they increase the surplus from trade with any retailer. Moreover, a retailer is unlikely to be perfectly informed about the cost of a supplier it randomly meets. Another example is the labor market. New graduates, even those with the same academic degree, have heterogeneous costs of labor, partly due to their different levels of costly training.

output at cost $x_0(\in (0, y_0))$ and can invest to reduce the production cost before entry. We denote the production cost resulting from the socially optimal investment as $x^*(\in (0, x_0))$.[2] After the sunk investments have been made, all agents (entrants and incumbents) randomly form one-buyer-to-one-seller pairs. Within each pair, the buyer makes a take-it-or-leave-it offer without observing the seller's investments. If the offer is accepted, then production takes place and both agents leave the market permanently. Otherwise, the pair is dissolved and both agents search in the next period. At the beginning of the next period, there are new buyers and sellers contemplate entering and the economy repeats the same matching and bargaining process. We assume that agents are impatient and the time between two successive periods is the source of search friction. In the model, agents are referred to as buyers and sellers. However, the agents can have more general roles such as firms and workers, retailers and intermediate suppliers, foreign importers and domestic exporters, etc.

This baseline model produces three main results. First, focusing on the steady-state equilibrium, the equilibrium exists and is unique. In equilibrium, ex ante identical seller entrants use mixed strategy when making investments and the resulting distribution of production costs has the support $[x^*, x_0]$. Ex ante identical buyers also use mixed strategy when offering prices and the resulting distribution of prices is non-degenerate.

Second, although the average amount of investments is positive, the agents' ex ante payoffs and the social welfare equal to the equilibrium values when investments are observable (indicating no investment) and are constant given any search friction. The positive investments create ex post gains. However, the corresponding ex ante gains depend on the speed of trade. In equilibrium, the ex post gains are completely dissipated by delays in trade for any given search friction, due to the possibility of a mismatch between the seller's reserve price and the buyer's offer. This result demonstrates that more efficient investments do not necessarily lead to a higher social welfare.

Third, we investigate how equilibrium distributions vary as search frictions di-

[2]The production cost x^* resulting from the socially efficient investments will be defined shortly in the next section.

minish and gather a set of empirically testable predictions. First, the investment strategy becomes more efficient in the sense that the proportion of entrants who invest efficiently increases to one. In contrast, the distribution of the incumbents' pre-entry investments becomes less efficient and converges to a point mass at zero investment. It is because the buyers price more aggressively as search frictions decrease. The relative trading speed between a seller with high and low investments enlarges and consequently the market accumulates more underinvested sellers, despite of the fact that more entrants invest efficiently.

In sum, this paper highlights the importance of the interaction between the pre-entry investments and trading outcomes. Notice that many of the above results would be drastically different if we separate the two. For instance, it is a robust result that the trading outcome becomes efficient as search frictions vanish if we treat the type distribution of the sellers as exogenously given. In addition, more efficient investments most likely improve the social welfare if the trading outcomes are held as fixed.

The baseline model is extended along two dimensions to incorporate the possibilities of two-sided investments and two-sided offers, respectively. In the first extension, buyer entrants could also invest before entry to raise their valuations of the good. The results of the baseline model regarding the sellers' investment strategy continue to hold. Moreover, we have a set of new results for the buyer's investment strategy. Most significantly, although the buyers have all of the bargaining power, they still underinvest and mix over an interval of investments given any search friction.

The model can be extended to allow sellers to make take-it-or-leave-it offers occasionally. The social welfare is still completely determined by the lowest investment level, which is now positive as a seller makes offer with a positive probability. Any welfare gain that could be generated from investments above the minimum level is dissipated by delays in trade. Fortunately, when sellers make offer with a probability bounded away from zero, the minimum investment level and the social welfare converge to the first best as search frictions vanish.

Related Literature

This paper is related to random search models with heterogeneous agents. Albrecht and Vroman (1992) demonstrate that when seller entrants are exogenously heterogeneous, a single price can never be an equilibrium. The current paper complements their paper by showing that such heterogeneity can emerge endogenously due to diverse prices in equilibrium.

The searching stage of the current model is similar to settings of voluminous works on search and bargaining games with asymmetric information (e.g., Rubinstein and Wolinsky (1990), Satterthwaite and Shneyerov (2007), Shneyerov and Wong (2010a), Lauermann (2012) (2013), etc). One central topic of the literature is to understand how frictions affect equilibrium efficiency. For instance, Lauermann (2013) shows that when there exists competitive pressure, equilibrium outcomes converge to perfect competition as search frictions disappear, indicating no delay in trade. On the contrary, delays in trade in the current setting is most severe as search frictions become arbitrarily small, and the social welfare in equilibrium is constant over any search friction. The reason for this drastically different prediction is the following. As search frictions vary in the current model, the distribution of entrants' production costs must change to ensure the buyers' indifference condition. Conversely, the distribution of entrant's production costs is fixed over all search frictions in other works. This comparison signifies the importance of the interaction between pre-entry investments and trade.

In addition, this paper is related to the literature on the hold-up problem. Acemoglu and Shimer (1999) also studies how investment incentives and trading outcomes are jointly determined when there is search friction. Unlike us, they assume that investments are observable, agents use Nash bargaining to split the surplus and entry is endogenous, which lead to quite different equilibrium outcomes. However, we share one common result: the conditions for efficient trading, the Hosios'condition in theirs and the arbitrarily small search frictions in ours, are no longer sufficient if entrants' characteristics are endogenous.

The investment incentives when investments are unobservable are also examined by Gul (2001) within a Coasian setting where a buyer's valuation is determined by his or her unobservable investments. Similar to ours, the equilibrium investment strategy is a mixed strategy and becomes efficient as the time between two rounds shrinks to zero. The difference is that in Gul (2001) there is no bargaining delay in

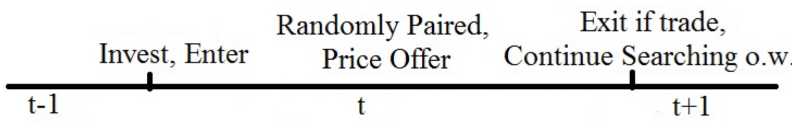

Figure 1: Timeline

the limit and the social welfare converges to the first best. This difference results from the distinct trading environments. We focus on big markets. Because the sellers who invest more trade faster, the investment distribution of incumbents is always less efficient than that of entrants. Hence, there are delays in trade even with almost efficient investments. On the other hand, Gul (2001) focuses on one-to-one trade. The investment distribution of the incumbent and entrant are by definition identical. As a result, as long as the incentive of efficient investments is guaranteed, the equilibrium outcome is socially optimal.

The rest of this paper is organized as follows. The model is introduced in Section 2, in which we also solve the first best and observable investment benchmark cases. Section 3 derives the equilibrium conditions and proofs the existence and uniqueness of the steady state equilibrium. The equilibrium is characterized in Section 4. Section 5 examines the two-sided investment extension, and Section 6 considers another extension with two-sided offers. Robustness and other extensions are discussed in Section 7. Finally, Section 8 concludes the paper.

2 The Model and Benchmark Specifications

2.1 The Model

We consider a discrete time infinite horizon random search model with pre-entry investments. Through out this paper, we focus on the steady state equilibrium. Therefore, we omit the time subscript whenever is not confusing. The timeline of this game is illustrated in Figure 1.

Player: The players are sellers and buyers. At the beginning of each period, one unit mass of new sellers and buyers enter the market. A buyer demands one unit of the output and yields utility $y_0 > 0$ from the consumption; a seller entrant is endowed with the technology that produces one unit of output at a cost $x_0 > 0$. We focus on the "gap" case throughout this paper, i.e., the (minimum) surplus from trade $y_0 - x_0$ is assumed to be strictly positive.

Strategy: Before entering the market, a seller can invest $c(x)$ to decrease the production cost to $x \geq 0$. We assume $c(x_0) = c'(x_0) = 0$, $c'(0) < -1$, and for any $x < x_0$, $c(x)$ is of class C^1, strictly decreasing and strictly convex. All of the entrants on both sides then join the incumbents who did not exit in the last period. The market sizes on both sides are assumed to be the same. In each period, one buyer is randomly matched with one seller and vice versa. The buyer in each pair makes a take-it-or-leave-it offer p, which is a monetary transfer from the buyer to the seller, and the seller decides whether to accept it.

Therefore, a seller's strategy consists of two components: an investment strategy governed by a CDF $F_e(x)$ and a reserve price mapping $r_S(x)$, where $F_e(x)$ measures the probability that the investment is weakly higher than $c(x)$ and $r_S(x)$ is the lowest price that a seller with cost x is willing to accept. A buyer's strategy is a price offer governed by a CDF $H(p)$, where $H(p)$ equals the probability of offering a price weakly lower than p.

Preference: If the offer is accepted, then one unit of output is produced and sold, which leaves the seller payoff $p - x$ and the buyer payoff $y_0 - p$. Both agents exit the market permanently. Otherwise, the pair is dissolved and both agents search in the next period.

The time between two successive periods is t and we assume that the discount rate for all agents are the same, which is denoted as r_1. Therefore, the discount factor is $\beta = e^{-r_1 t} \in [0, 1)$. We say that the search frictions are small if t is small or equivalently if β is large.

Information: A crucial assumption of this paper is that the buyers have no information about investments. In addition, the matching is anonymous.

2.2 Benchmark Specifications

The First Best

We first characterize the efficient allocation, which consists of both the efficient investment and trade.

At the search stage, a social planner would find it optimal to always conduct trade between any two agents given any cost distribution, as the surplus from the trade is always positive and postponing trading is costly due to discounting.

Given that trades take place immediately at the search stage, if a seller invests to decrease his or her cost to x, then he or she increases the social welfare by $x_0 - x$ with an investment cost $c(x)$. A social planner thus chooses x^* implicitly defined by $c'(x^*) = -1$ so that the marginal cost of investment equals the marginal benefit. From the assumptions on $c(x)$, it is easy to verify that $x^* \in (0, x_0)$.

Observable Investments

Consider the situation in which investments are observable. Following the same logic used in Diamond (1971), as the buyers have all of the bargaining power, a seller receives zero search stage payoff regardless of his or her production cost. The reasoning is as follows. The buyer in the current match and buyers in al-l future matches will offer exactly the seller's production cost plus the discounted continuation payoff, as the buyers can observe the seller's production cost. With the discount factor being strictly less than one, this infinitely repeated discounting drives the search stage payoff down to zero. The sellers therefore have no incentive to invest.

Therefore, in the unique equilibrium, no seller invests and all of the buyers offer a price x_0. Investments are inefficient and trades are efficient.

This analysis shows why unobservability is necessary to restore investment incentives. When buyers cannot observe (or perfectly infer in equilibrium) investments, a seller may receive a price offer larger than his or her reservation price. Such a possibility creates rents for sellers. We illustrate this intuition in detail in the next section.

3 The Steady State Equilibrium

Let us now solve the steady state equilibrium in the decentralized market. A steady state equilibrium consists of a seller's investment strategy when he or she is an entrant and reserve price function when he or she is an incumbent, a buyer's price offer distribution and a distribution of incumbents' production costs. Following the literature, we will call the last distribution as stationary cost distribution.

3.1 The Seller's Problem

At the search stage, a seller with a production cost x chooses the lowest price he or she is willing to accept, i.e., the reserve price $r_S(x)$, to maximize his or her search stage payoff $U(x)$. Given a price offer distribution $H(p)$, the seller's trading probability is $1 - H(r_S(x)) + Pr(p = r_S(x))$, which is decreasing in $r_S(x)$. Here, $Pr(p = r_S(x))$ is the probability of a price offer that equals $r_S(x)$ according to $H(p)$. The maximization problem of a type x seller can be summarized as follows:

$$U(x) = \max_r \{(E(p \mid p \geq r) - x)(1 - H(r) + Pr(p = r)) + (H(r) - Pr(p = r))\beta U(x)\} \tag{1}$$

Solving the above problem, the reserve price $r_S(x)$ can be calculated as follows:

$$r_S(x) = x + \beta U(x) \tag{2}$$

A seller is willing to accept any price that is high enough to cover his or her opportunity cost of trading, i.e., the production cost x plus the discounted continuation payoff.

A more efficient seller should have a higher search stage payoff $U(x)$, as investments are costly. This more efficient seller is willing to accept lower price offers in equilibrium, as the opportunity cost of delay is higher. Moreover, the least efficient seller should receive zero search stage payoff. No buyer in equilibrium would offer prices higher than his or her reserve price. Then the insights in Diamond Paradox implies that the least efficient seller should receive zero search stage payoff. Denote the highest production cost on the support as $\bar{x}$. The following lemma confirms the preceding conjectures.

Lemma 1. *In any steady state equilibrium, $U(x)$ is strictly decreasing and continuous in x, with $U(\bar{x}) = 0$. $r_S(x)$ is strictly increasing and continuous in x.*

Unless otherwise mentioned, all of the proofs are gathered in the Appendices.

Lemma 1 implies that $\hat{x}(p)$, the inverse function of $r_S(x)$, is well defined, continuous and strictly increasing. Function $\hat{x}(p)$ specifies the highest type of a seller who is willing to accept a price p.

Lemma 1 also implies that the highest cost on the support equals the initial cost, i.e. $\bar{x} = x_0$. A seller with the highest cost is fully extracted in the search stage and therefore has no incentive to invest ex ante. In other words, the holdup problem holds for the least efficient sellers, although their investments are unobservable and they can search for other buyers.

Corollary 1. *In any steady state equilibrium, the least efficient sellers invest zero,* $\bar{x} = x_0$.

The least efficient sellers' ex ante payoff therefore equals $U(\bar{x}) - c(x_0) = 0$. Recall that the investment strategy is governed by $F_e(x)$. In equilibrium, ex ante identical sellers must be indifferent over any x on the support of F_e, and weakly prefer these x to any other x that is not on the support. As can be seen from (1), the search stage payoff $U(x)$ depends on the price offer distribution $H(p)$. In equilibrium, $H(p)$ must be such that

$$U(x) - c(x) = 0, \text{ for any } x \text{ on the support of } F_e(x) \tag{3}$$
$$U(x) - c(x) \leq 0, \text{ for any } x \text{ not on the support of } F_e(x)$$

3.2 The Buyer's Problem

A buyer chooses what price to offer. We know from the previous section that any seller with a cost lower than $\hat{x}(p)$ is willing to accept p. We use $F(x)$ to denote the CDF of the stationary cost distribution, which equals the probability that the production cost of a randomly drawn incumbent is weakly lower than x. The probability of trade therefore equals $F(\hat{x}(p))$ for price p.

In equilibrium, $F(x)$ must be such that it makes a buyer indifferent over any p on the support of $H(p)$. We use π to denote the equilibrium payoff of a buyer. Then any p on the support solves the following maximization problem:

$$\pi = \max_{p}\{(y_0 - p)F(\hat{x}(p)) + (1 - F(\hat{x}(p)))\beta\pi\} \tag{4}$$

3.3 The Seller's Investment Strategy

The last piece of the model is the distribution of seller entrants' production cost, which is also the seller's investment strategy.[3] In a steady state equilibrium, the measure of outflow of any type must equal the measure of inflow of the same type to preserve the stationary distribution over time. A seller with a cost x leaves the market if he or she receives an offer that is weakly higher than $r_S(x)$ (which happens with a probability $1 - H(r_S(x))$).[4] Meanwhile, the measure of entrants with a cost lower than x is $F_e(x)$. Denote the lowest production cost on the support as $\underline{x}$. The steady state equilibrium requires that for any x on the support,

$$F_e(x) = \frac{F(x) - \int_{\underline{x}}^{x} H(r_S(\tilde{x}))dF(\tilde{x})}{1 - \int_{\underline{x}}^{x_0} H(r_S(\tilde{x}))dF(\tilde{x})} \tag{5}$$

3.4 Equilibrium Existence and Uniqueness

Let us first summarize the dynamic of a steady state equilibrium. At any period, the stationary cost distribution $F(x)$ is such that it keeps buyers indifferent to price offers. The players' trading strategy then determines the cost distribution of those who exit. At the beginning of the next period, the new generation of seller entrants, who are indifferent to these investments given the price distribution $H(p)$, choose the investments so that they exactly replace those who exit. This way, the stationary cost distribution is preserved over time.

Proposition 1. *In any steady state equilibrium, the price offer distribution $H(p)$, the seller's investment strategy $F_e(x)$ and the stationary cost distribution $F(x)$ have*

[3]As there is one unit mass of entrants, we obtain the equivalence between the distribution of entrants' costs and the investment strategy when we abuse the law of large numbers as usual.

[4]To be more precise, the probability of receiving an offer weakly higher than $r_S(x)$ should be $1 - H(r_S(x)) + Pr(p = r_S(x))$. However, as is proved in the next section, there is no mass point on the support of $H(p)$. To simplify the notation, we write down the probability here as if there is no mass point.

the following properties:

1. *$F(x)$ and $F_e(x)$ have support $[x^*, x_0]$ with the unique point mass at x^*.*

2. *$H(p)$ has support $[r_S(x^*), r_S(x_0)]$ and is atomless;*

Although all agents are identical ex ante, proposition 1 shows that the stationary price offer distribution and the investment strategy are non-degenerate. The unobservability of the investments is the key behind this result. Suppose otherwise, i.e., that all buyers offer the same price. The sellers become residual claimants facing the single price, as the unobservability of investments prevents any further exploitation. They either invest efficiently or invest zero. If the price is not high enough to cover the investment cost plus the production cost ($p < c(x^*) + x^*$), then the sellers invest zero. In addition, because $c(x^*) + x^* < x_0$, the price is lower than the initial production cost. As a result, the sellers reject the offer after entering the market. This leaves the buyers with zero profit. Therefore, to earn a positive profit, the buyers must offer a price $\hat{p}$ higher than $c(x^*) + x^*$ so that the sellers invest efficiently ex ante and agree to accept the price. However, given that all of the other buyers offer the price $\hat{p}$, a buyer would find it profitable to deviate to a slightly lower price. Due to discounting, the matched seller is still willing to accept the lower price. Hence, a single price can never be an equilibrium. To support such dispersed price distribution, the investment distribution also has to be non-degenerate.

We may also suspect a gap on the support of the price and stationary cost distributions. For instance, it may be the case that two prices p_1 and p_2 are on the support, but the prices on the interval (p_1, p_2) are not offered because the sellers with types $x \in (\hat{x}(p_1), \hat{x}(p_2))$ are not in the market and hence the buyers do not gain from offering their reserve prices. Meanwhile, no sellers choose to become type $x \in (\hat{x}(p_1), \hat{x}(p_2))$ because their reserve prices are not offered. This intuition unfortunately neglects the indifference condition: a seller must be indifferent between $\hat{x}(p_1)$ and $\hat{x}(p_2)$ and weakly prefer them to any x in the interval. As no price between p_1 and p_2 is offered, $U(x)$ increases linearly on the interval $(\hat{x}(p_1), \hat{x}(p_2))$. However, the investment cost function $c(x)$ is strictly convex. Hence, the indifference condition can never hold with a gap.

Proposition 1 also claims that there is no mass point on the price distribution. Any mass point results in a jump in the probability of trade, which in turn leads to a kink in $U(x)$. However, the investment cost function $c(x)$ is of class C^1. This again contradicts the indifference condition.

Of course, the preceding reasoning is based on the assumption that the investment cost function $c(x)$ is of class C^1, continuous and strictly convex. If $c(x)$ is not so well behaved, then some properties of the price distribution change accordingly.

This proposition also shows that the lowest production cost on the market is the efficient cost x^*. The seller with the lowest cost trades immediately with a probability of one: any price offer in the market is weakly higher than his or her reserve price. As the seller's investments are unobservable to the buyer, he or she becomes the residual claimant and invests efficiently.

Finally, there is a mass point at x^* in both $F_e(x)$ and $F(x)$, as a buyer who offers $r_S(x^*)$ can only trade with these sellers and must get strictly positive equilibrium payoff. Moreover, there is no other mass point, as any of such point would lead to a jump in the buyer's payoff as a function of p. This contradicts the buyer's indifference condition.

The preceding results and intuitions hold even when $\beta = 0$, in which case the buyer in each pair is a monopolist. The monopolist is indifferent to the price interval because the stationary cost distribution is adjusted so that the demand function is unit elastic at any price in the interval.

In the rest of this section, we solve $H(p)$ and $F(x)$ and then show the existence and uniqueness of the steady state equilibrium.

$H(p)$ can be solved from the envelope condition of $U(x)$. It is legitimate to take the derivative of $U(x)$ because we have proved that the support is an interval and that $U(x) - c(x) = 0$ for any x on the support. $c(x)$ is of class C^1, which implies that $U(x)$ is also of class C^1. The envelope condition is

$$U'(x) = -(1 - H(r_S(x))) + H(r_S(x))\beta U'(x)$$

Using the equilibrium restriction that $U'(x) = c'(x)$, $H(p)$ can be solved:

$$H(p) = \frac{1 + c'(\hat{x}(p))}{1 + \beta c'(\hat{x}(p))} \tag{6}$$

$F(x)$ is solved from the buyer's indifference condition. If a buyer offers the highest reserve price $r_S(x_0) = x_0 + \beta U(x_0) = x_0$, he or she can trade with a probability of one. Therefore, $\pi = y_0 - x_0$.

Any other price on the support must yield the same expected profit. In other words,

$$(y_0 - p)F(\hat{x}(p)) + [1 - F(\hat{x}(p))]\beta\pi = y_0 - x_0$$

Therefore, the stationary cost distribution $F(x)$ can be calculated as follows:

$$F(x) = \begin{cases} 0, \text{ if } x \in (-\infty, x^*), \\ \frac{y_0 - \beta\pi - x_0}{y_0 - \beta\pi - x - \beta c(x)}, \text{ if } x \in [x^*, x_0], \\ 1, \text{ if } x \in (x_0, +\infty). \end{cases} \tag{7}$$

We have shown that $F(x)$ and $H(p)$ exist and are unique. Thus, the investment strategy $F_e(x)$ also exists and is uniquely determined by (5).

Proposition 2. *The steady state equilibrium exists and is unique.*

4 Equilibrium Characterization

4.1 Investment Strategy and Stationary Cost Distribution

There are two distributions of sellers' type. One is the investment strategy $F_e(x)$, which is the cost distribution of entrants. The other is the stationary cost distribution $F(x)$, which is the cost distribution of incumbents. We have already shown that the per-period trading probability increases in the investment level, or equivalently decreases in the production cost. Therefore, a more efficient seller trades and exits the market more rapidly. In other words, the cost distribution of sellers who exit is more efficient than the stationary cost distribution. This in turn implies that the investment strategy $F_e(x)$ is always more efficient than $F(x)$.

Proposition 3. *The stationary cost distribution $F(x)$ has first order stochastic dominance over the investment strategy $F_e(x)$.*

4.2 Constant Payoffs and Social Welfare

As shown in the previous section, sellers always invest with a positive probability given any search friction. The social welfare is expected to be higher than that obtained in the benchmark case with observable investments, where sellers have no incentive to invest. However, this is not the case as shown in the following theorem.

Theorem 1. *For any $\beta \in [0, 1)$, the seller's ex ante payoff is 0, and the buyer's ex ante payoff equals the social welfare, which is $y_0 - x_0$.*

We can easily verify that the agents' payoffs and social welfare are the same in both observable and unobservable cases. Although the unobservability incentivizes investments, it also causes trading inefficiency. The welfare gain generated from investments can be realized fully only if the seller and buyer entrants agree to trade immediately after entering the market. However, this is impossible given the presence of information and search frictions. Due to the unobservability, both the cost and the price distributions are non-degenerate. Profitable trades are therefore conducted only probabilistically. In other words, in expectation there is a expected delay in trade for any buyer whose price offer is strictly lower than $r_S(x_0)$. The welfare loss due to the delay in trade exactly offsets the welfare gain from the more efficient investments.

The social welfare remains constant over the search frictions. As the social welfare depends on both the investment and trade efficiency, the constant welfare result may arise from the constant investment and trade efficiency, from more efficient investments and less efficient trade or vice versa. To determine which is the case, in the next section we investigate the change in investment strategy and efficiency of trade as we vary the search frictions.

4.3 Comparative Statics and the Limiting Case

In this paper, the search friction is captured by the fact that it takes time t to meet with the next potential partner. In this section, we illustrate how equilibrium outcomes vary as search frictions change, or equivalently, as t and β changes. It is easy to see that β strictly decreases in t.

Consider a seller with some cost $x \in (x^*, x_0)$. As meetings become more frequent, the seller trades with a higher probability per unit of time if the price distribution remains constant. Consequently, the new marginal benefit of investment, which strictly increases in the probability of trade per unit of time, is strictly larger than the original marginal cost of investment. Therefore, to keep a seller indifferent across investments when the search frictions decrease, the buyers must price more aggressively. That is, the per-period trading probability $1 - H(r_S(x))$ must strictly decrease in β. Indeed,

$$\frac{\partial(1 - H(r_S(x)))}{\partial\beta} = \frac{(1 + c'(x))c'(x)}{(1 + \beta c'(x))^2} < 0, \text{ for any } x \in (x^*, x_0)$$

As the search frictions vanish, the probability of trade per-period $1 - H(r_S(x))$ must converge to zero for any $x \in (x^*, x_0)$. Equivalently, buyers must price extremely aggressively in the limit, i.e., the price offer distribution must converge in distribution to a point mass at $r_S(x^*)$. Otherwise, any seller trades almost certainly within any small amount of time. The marginal benefit of investment therefore becomes one and a seller cannot be indifferent across investment levels. We can also mathematically verify this intuition from (6) by taking the limit $\beta \to 1$.

The next result is that the stationary cost distribution $F(x)$ becomes less efficient as β increases. If $F(x)$ stays constant when meetings become more frequent, a buyer who is originally indifferent over price offers strictly prefers to offer the lowest price $r_S(x^*)$. Therefore, the per-period trading probability of a price lower than $r_S(x_0)$ must decrease in β. That is, the stationary cost distribution $F(x)$ with a larger β has first order stochastic dominance over an $F(x)$ with a smaller β. Indeed,

$$\frac{\partial F(x)}{\partial\beta} = \frac{(y_0 - x_0)(x - x_0 + c(x))}{[y_0 - \beta\pi - x - \beta c(x)]^2} < 0$$

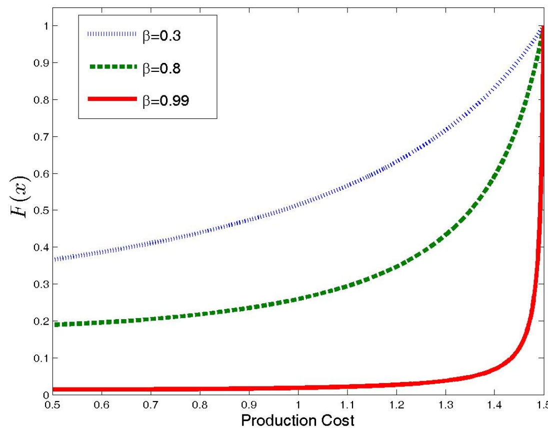

Figure 2: Stationary Seller Type Distribution
(*In this example:* $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$.)

In the limit, we can verify that $F(x)$ converges in distribution to a point mass at x_0 from (7). That is, almost all of the incumbents are composed of sellers who invested zero. This result is shown graphically in Figure 2. If $F(x)$ is bounded away from 0 in the limit for some $x < x_0$, then the per-period trading probability of a buyer offering $r_S(x)$ is strictly positive and the buyer can trade immediately as the time between two successive meetings shrinks to zero. Hence, a buyer who offers $r_S(x_0)$ would find it optimal to lower the price offer to $r_S(x)$ without changing the trading probability, leading to a contradiction.

In the steady state, the cost distribution of entrants is the same as that of exits to preserve the stationary cost distribution. As the cost distribution of incumbents becomes less efficient as β increases, the distribution of exits and hence the investment strategy $F_e(x)$ may become less efficient.

It turns out that the conjecture is not quite right. We find that the point mass $F_e(x^*)$ strictly increases in β, i.e., an entrant is more likely to invest efficiently. In the limit, $F_e(x^*)$ increases to one and hence the investment strategy becomes efficient. We already know that $F_e(x)$ is always more efficient than $F(x)$. Our limiting result further shows that the two distributions converge to two polar points, respectively. The investment strategy with different discount factors is plotted in Figure 3.

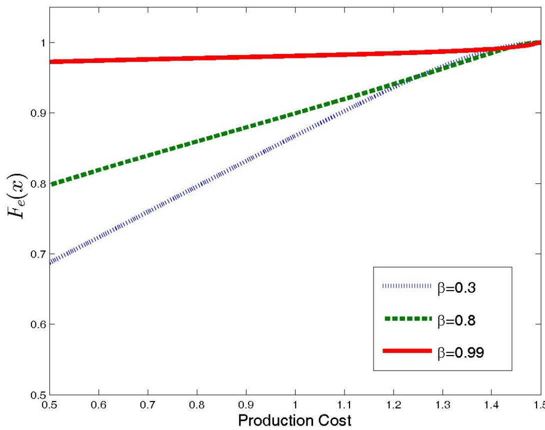

Figure 3: Seller's Investment Strategy
(*In this example:* $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$.)

To understand this result, note that a seller who invests efficiently always exits the market immediately independent of β. In addition, we have three observations from previous analysis: as β increases, 1) a larger proportion of the incumbents are the underinvested type; 2) the per-period trading probabilities of underinvested sellers strictly decrease; and 3) the average cost of underinvested sellers strictly increases, which implies that a seller with the average cost is less likely to trade. The first effect raises the proportion of underinvested exits, as captured by the previously mentioned casual intuition. The remaining two effects explain why the proportion of underinvested exits decreases in β: any underinvested seller is less likely to trade and the composition of the underinvested sellers becomes less efficient. The latter dominates the former.

We summarize the preceding comparative statics and limiting results in the following theorem.

Theorem 2. *As β increases to one, in the steady state equilibrium*

1. *$H(r_S(x))$ strictly increases for any $x \in (x^*, x_0)$ and converges in distribution to a point mass at $r_S(x^*)$;*

2. *$F(x)$ strictly decreases for any $x \in [x^*, x_0)$ and converges in distribution to a point mass at x_0; and*

3. *$F_e(x^*)$ strictly increases and $F_e(x)$ converges in distribution to a point mass at x^*.*

From this theorem, we can better understand the mechanism behind the constant social welfare result. As β increases, more of the new entrants invest efficiently, and this may generate additional social welfare if the trading efficiency remains constant. Unfortunately, trades become less efficient at the same time. As β increases, the stationary cost distribution is more concentrated on higher costs. Consequently, it takes more periods for a buyer offering a given price to trade.

4.4 Convergence from the initial time to the steady state

So far we have focused on the steady state equilibrium. In this section, we investigate the existence of a path starting from date zero and converging to the steady state, such that the equilibrium properties in the steady state are preserved along the path.

To construct such a path, we assume that agents in any cohort (correctly) expect to receive the same search stage payoffs as in the steady state equilibrium. In period $t = 1$, we let the seller entrants to invest according to $F(x)$ and the buyers to price according to $H(p)$. Seller entrants obey the order because they are indifferent to any investment level on the support given their expected search stage payoffs. Buyers also obey, as the cost distribution of incumbents at $t = 1$ is $F(x)$. At the end of date $t = 1$, only some of the agents (denoted by $e_1 < 1$) trade and exit, and the cost distribution of exiting sellers is exactly $F_e(x)$.

At the beginning of date $t = 2$, the new cohort of sellers are again indifferent to the investment levels. Let measure e_1 of them invest according to $F_e(x)$ and measure $1 - e_1$ of them invest according to $F(x)$. This way, the incumbents' cost distribution at $t = 2$ remains $F(x)$. The price distribution of buyers is therefore the same. Furthermore, the size of exits at $t = 2$, e_2, remains smaller than one.

In general, at any date along the path, the size of the exits is always smaller than the size of the entrants. Therefore, the investment distribution of entrants in any cohort can be constructed as previously specified to preserve $F(x)$. The size of the market will eventually grow to the size in steady state and the size of the exits will be equal to that of the entrants. In the end the equilibrium converges to the steady state.

As the preceding construction shows, the agents' ex ante payoffs, social welfare, incumbents' cost distribution and price distribution are constant along the path.

Proposition 4. *There exists an equilibrium path starting from the initial time and converging to the steady state such that*

1. *at any point along the path, the incumbents' cost distribution is $F(x)$ as specified in (7) and the price distribution is $H(p)$ as specified in (6);*

2. *within any cohort, the seller's ex ante payoff is zero, the buyer's ex-ante payoff is $y_0 - x_0$ and the social welfare is $y_0 - x_0$; and*

3. *the investment strategy becomes more efficient as time increases.*

The rest of the paper extends the baseline model along two directions, respectively. Section 5 considers the situation in which a buyer may invest to raise the valuation. Section 6 examines a two-sided offering case in which a seller makes a take-it-or-leave-it offer with a positive probability.

5 Two-Sided Investments

In many cases, buyers can also invest before searching for sellers. For instance, a firm can invest in technology to raise the output per unit of labor. The main message from studying this extension is that the buyers in this environment underinvest even if they have all of the bargaining power.

We assume that before entering, a buyer can increase the value from y_0 to y with investments $e(y)$, where $e(y_0) = e'(y_0) = 0$ and $e(y)$ is strictly increasing and strictly convex for any $y > y_0$. The surplus from trade between x and y is $y - x$. This assumption implies no complementarity between investments, which greatly simplifies the analysis[5]. Finally, the observability of the buyer's investments may

[5]In the case with one-sided investment, we can make this assumption without a loss of generality, as all of the buyers are identical. In the two-sided investment case, however, this assumption

be arbitrary, as there is no complementarity by assumption and buyers have all of the bargaining power.

Consider the benchmark cases of a) the first best and b) observable investments. In the first best benchmark case, as there is no complementarity, the social planner commends all agents to trade upon their first meetings. Given this, the planner asks all sellers to reduce the production cost to x^* and all buyers to raise the value to y^*, where x^* is defined as before and y^* is uniquely determined by $e'(y^*) = 1$.

In the benchmark case with observable investments, all sellers invest zero and all buyers raise the valuation to y^* and offer a price x_0. In equilibrium, a seller receives a payoff of zero and a buyer receives a payoff of $y^* - x_0 - e(y^*)$, which is also the social welfare.

In the rest of this section, we derive the optimality conditions of the steady state equilibrium and then characterize the equilibrium.

5.1 The Steady State Equilibrium

Due to the lack of complementarity, sellers cannot benefit from the buyers' investments directly and care only about the price distribution. Therefore, the seller's problem is exactly the same as before and the previous equilibrium conditions for sellers continue to hold.

We therefore focus on the buyer's problem in the rest of this section. A buyer's strategy consists of his or her investment strategy $G_e(y)$ and the price offer $p(y)$. $p(y)$ maximizes the search stage payoff denoted as $\Pi(y)$,

$$\Pi(y) = \max_p \{(y-p)F(\hat{x}(p)) + [1 - F(\hat{x}(p))]\beta\Pi(y)\} \tag{8}$$

Moreover, the following indifference conditions must hold.

$$\begin{aligned} &\Pi(y) - e(y) = \pi \geq 0, \text{ for any } y \text{ on the support of } G_e(y) \\ &\Pi(y) - e(y) \leq \pi, \text{ for any } y \text{ not on the support of } G_e(y) \end{aligned} \tag{9}$$

excludes some interesting scenarios. For instance, the surplus from trade could be supermodular, in which case a higher seller's investment level leads to a larger marginal benefit of the buyer's investment. We will leave the analysis for more general cases as future work.

They essentially require a buyer to be indifferent across any y on the support of the investment strategy $G_e(y)$ and weakly prefer those y to any other y that is not on the support. The difference between the search stage payoff and the investment cost π is the buyer's ex ante payoff.

Lemma 2. *In any steady state equilibrium with two-sided investments, $p(y)$ is single-valued and increases in y for any y on the support of $G_e(y)$.*

Lemma 2 shows that the price offer increases in buyer's type. The waiting cost of a buyer with a higher valuation is larger. The buyer is willing to offer a higher price to ensure a higher trading probability. Consequently, the buyer with the highest valuation on the support, denoted as $\bar{y}$, offers the highest price, which equals the reserve price of a type x_0 seller, i.e., $p(\bar{y}) = x_0$ and $F(\hat{x}(p(\bar{y}))) = 1$. Therefore, $\Pi(\bar{y}) = \bar{y} - x_0$ and $\pi = \Pi(\bar{y}) - e(\bar{y}) = \bar{y} - x_0 - e(\bar{y}) > 0$.

Lemma 2 also demonstrates that given his or her ex ante investments, a buyer will never play a mixed pricing strategy at the search stage. Suppose that there are two buyers investing the same $e(y)$ but offering different prices. In particular, buyer 1 offers price p_1 and buyer 2 offers price $p_2 < p_1$. Given the non-degenerate production cost distribution, buyer 1 trades faster in expectation and therefore has a larger marginal benefit of investment. However, as they choose the same investment level, the marginal cost of investment is the same for both, leading to a contradiction.

More importantly, the pure pricing strategy implies that although the buyers have all of the bargaining power, they adopt a mixed investment strategy and hence underinvest with a strictly positive probability.

Proposition 5. *In any steady state equilibrium with two-sided investments, the seller's investment strategy $F_e(x)$, stationary cost distribution $F(x)$, buyer's investment strategy $G_e(y)$ and stationary valuation distribution $G(y)$ have the following properties:*

1. *$F_e(x)$ and $F(x)$ have support $[x^*, x_0]$ with the unique point mass at x^*; and*

2. *$G_e(y)$ and $G(y)$ have support $[\underline{y}, y^*]$ and are atomless, where $\underline{y}$ is uniquely determined by*

$$y^* - x_0 - e(y^*) = [\underline{y} - x^* - \beta c(x^*)]e'(\underline{y}) - e(\underline{y}) \tag{10}$$

Finally, the steady state condition must hold for buyers. That is, the investment strategy $G_e(y)$ must equal the valuation distribution of the buyers who exit the market. A buyer exits when his or her offer is accepted, which happens with a probability of $F(\hat{x}(p(y)))$. Combined with $G(y)$, the distribution of exits is determined. Equating the entrant and exit distributions, we have the following equilibrium condition.

$$G_e(y) = \frac{\int_{\underline{y}}^{y} F(\hat{x}(p(\tilde{y})))dG(\tilde{y})}{\int_{\underline{y}}^{y^*} F(\hat{x}(p(\tilde{y})))dG(\tilde{y})} \tag{11}$$

We are now ready to solve the equilibrium. The convex supports and the indifference conditions imply that both $U(x)$ and $\Pi(y)$ are differentiable. We can therefore use the envelope conditions to solve for the stationary distributions and the price offer function $p(y)$. The derivation also proves the existence and uniqueness of the steady state equilibrium. We summarize the results in the following lemma.

Lemma 3. *The steady state equilibrium with two-sided investments exists. The stationary cost distribution CDF $F(x)$ is defined by (13), the sellers' investment strategy CDF $F_e(x)$ is defined by (5), the reserve price $r_S(x)$ is defined by (2), the stationary valuation distribution CDF $G(y)$ is defined by (14), the buyers' investment strategy CDF is defined by (11) and the price offer $p(y)$ is defined by (12).*

Moreover, the steady state equilibrium is unique.

$$p(y) = y - \frac{e(y) + y^* - x_0 - e(y^*)}{e'(y)} \tag{12}$$

$$F(x) = \begin{cases} 0, \textit{ if } x \in (-\infty, x^*), \\ \frac{(1-\beta)e'(\hat{y}(r_S(x)))}{1-\beta e'(\hat{y}(r_S(x)))}, \textit{ if } x \in [x^*, x_0], \textit{ where } \hat{y}(p) \textit{ is the inverse of } p(y) \\ 1, \textit{ if } x \in (x_0, +\infty). \end{cases} \tag{13}$$

$$G(y) = \begin{cases} 0, \textit{ if } y \in (-\infty, \underline{y}) \\ \frac{1+c'(\hat{x}(p(y)))}{1+\beta c'(\hat{x}(p(y)))}, \textit{ if } y \in [\underline{y}, y^*], \\ 1, \textit{ if } y \in (y^*, +\infty). \end{cases} \tag{14}$$

In the baseline model, we have demonstrated that the equilibrium payoffs and social welfare are the same as if investments were observable. This result still holds in the two-sided investments extension.

Theorem 3. *In the steady state equilibrium with two-sided investments, the seller's ex ante payoff equals zero, the buyer's ex ante payoff and the social welfare equals* $y^* - x_0 - e(y^*)$.

5.2 Comparative Statics and the Limiting Case

The comparative statics and limiting results for $H(p)$, $F(x)$ and $F_e(x)$ in theorem 2 can be extended here. Therefore, we do not repeat the results but instead devote this section to the comparative statics exercise with the buyers' investment strategy $G_e(y)$ and stationary valuation distribution $G(y)$.

First, the lower bound $\underline{y}$ as defined in condition (10) strictly increases in β. To understand this result, we know that $\underline{y}$ offers the lowest reserve price $x^* + \beta c(x^*)$, which strictly increases in β. Meanwhile, the trading probability is strictly lower, as $F(x^*)$ strictly decreases in β. Therefore, $\underline{y}$ must strictly increase in β to keep the buyer's ex ante payoff constant ($\pi = y^* - x_0 - e(y^*)$).

Furthermore, the limit of $\underline{y}$ as $\beta \to 1$ is strictly less than y^*. In other words, the investment strategy is non-degenerate even when the search frictions vanish.

The stationary valuation distribution $G(y)$ also adjusts as β changes. We know from the baseline model that the price distribution concentrates more on lower prices as β increases. Given that $p(y)$ strictly increases in y, it must be the case that $G(y)$ concentrates more on smaller valuations. $G(y)$ indeed converges in distribution to a point mass at $\underline{y}$ in the limit. Figure 4 graphically shows the preceding two results.

The buyers' investment strategy $G_e(y)$ also converges in distribution to a point mass at $\underline{y}$ as β approaches one. We know that almost all of the buyers offer the lowest price in the limit. Therefore, almost all of the trades take place at the lowest price offered by $\underline{y}$. Hence, the entrants who replace these exits in the steady state consist almost entirely of type $\underline{y}$ buyers. The investment strategy with the same set

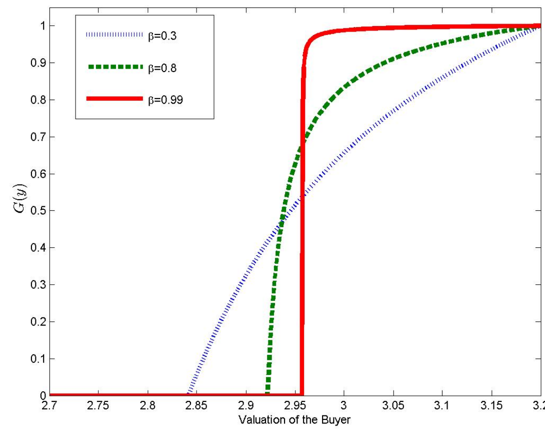

Figure 4: Stationary Buyer Type Distribution
(*In this example:* $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$, $e(y) = \frac{1}{2}(y - y_0)^2$.)

of parameters is plotted in Figure 5. [6]

The preceding discussions are summarized in proposition (6).

Proposition 6. *In the steady state equilibrium with two-sided investments,*

1. *as β increases to one, (i) the lowest valuation $\underline{y}$ strictly increases and (ii) a buyer with a valuation that equals the $t * 100$th percentile of $G(y)$ offers the reserve price of a more efficient seller, for any $t \in (0, 1)$; and*

2. *as $\beta \to 1$, (i) $\underline{y}$ is still bounded away from y^*, i.e., $\lim_{\beta \to 1} \underline{y} < y^*$, and (ii) $G_e(y)$ and $G(y)$ converge in distribution to a point mass at $\underline{y}$.*

6 Two-Sided Offers

In some situations, a seller also has the opportunity to make offers. In each meeting, assume nature randomly selects the seller to make a take-it-or-leave-it offer with a probability α that is bounded away from zero and selects the buyer with the

[6] To clarify, although the investment strategy concentrates more on lower valuations, we cannot conclude that the investment strategy becomes less efficient because $\underline{y}$ strictly increases in β.

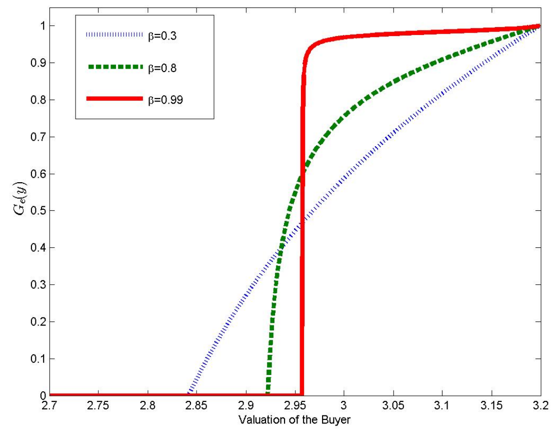

Figure 5: Buyer's Investment Strategy
(*In this example:* $x_0 = 1.5$, $c(x) = \frac{1}{2}(x - x_0)^2$, $y_0 = 2.2$, $e(y) = \frac{1}{2}(y - y_0)^2$.)

complementary probability. Therefore, a seller's strategy also includes a price offer $p_S(x)$, and a buyer's strategy also includes a reserve price r_R.

We can verify that $r_R = y_0 - \beta\pi$, i.e., a buyer is willing to pay the price if it leaves at least the discounted continuation payoff. Therefore, all sellers will propose $p_S(x) = r_R$ if $r_R - x$ is weakly higher than $\beta U(x)$.

6.1 Benchmark Case: Observable Investments

As a benchmark, let us first characterize the steady state equilibrium when investments are observable. It will become clear shortly that we can borrow a lot of the results in this benchmark for the analysis of the unobservable case.

First, as long as a positive amount is invested, a seller will choose $\bar{x}$ to maximize his or her ex ante payoff:

$$\bar{x} = argmax_x \{ \frac{\alpha(y_0 - x - \beta\pi)}{1 - \beta(1-\alpha)} - c(x) \}$$

Therefore, $\bar{x}$ is uniquely defined by

$$c'(\bar{x}) = \frac{-\alpha}{1 - \beta(1-\alpha)} \tag{15}$$

Lemma 4. *In any stationary equilibrium with two-sided offers and observable investments, either all of the incumbents have cost $\bar{x}$ and trade immediately, or some of the incumbents have cost x_0 and never trade. In both cases, all of the seller entrants invest to have cost $\bar{x}$.*

Therefore, depending on the parameters, one of the following two equilibria will arise. In the first type of equilibrium, the entrants in any period invest to become type $\bar{x}$ and are the sole source of incumbents. In other words, the market size equals the entrant size. All of the agents trade immediately. We know that a seller has an alternative option, which is to invest zero and receive zero payoff. Therefore, this equilibrium requires the search stage payoff, which depends on the bargaining power α and the shape of $c(x)$, to be larger than the investment cost.

When this condition fails to hold, we have the second type of equilibrium, in which the incumbents consist of not only the new entrants, but also some sellers who invest zero. All of the non-invested incumbents never trade and stay in the market forever. All of the invested incumbents trade immediately and are replaced by entrants in the next period. This equilibrium requires seller entrants to be indifferent over investing zero and $c(\bar{x})$. One way to understand the presence of these non-invested incumbents is to use the convergence analysis in section 4.4. Before reaching the steady state, entrants mix between investing and not investing, and the non-investing sellers never leave the market.

Lemma 5. *The steady state equilibrium with two-sided offers and observable investments takes one of the following two forms.*

1. *If $\alpha(y_0 - \bar{x}) \geq c(\bar{x})$, then both $F(x)$ and $F_e(x)$ are point masses at $\bar{x}$. The seller's ex ante payoff is $\alpha(y_0 - \bar{x}) - c(\bar{x})$ and the buyer's ex ante payoff is $(1 - \alpha)(y_0 - \bar{x})$.*

2. *If $\alpha(y_0 - \bar{x}) < c(\bar{x})$, then $F_e(x)$ is a point mass at $\bar{x}$ and $F(x)$ has two point masses at x_0 and $\bar{x}$, with $F(\bar{x}) = \frac{\alpha(y_0-\bar{x})}{\beta(1-\alpha)c(\bar{x})} - \frac{1-\beta(1-\alpha)}{\beta(1-\alpha)}$. The seller's ex ante payoff is zero and the buyer's ex ante payoff is $\frac{\alpha(y_0-\bar{x})-(1+\alpha\beta-\beta)c(\bar{x})}{\alpha\beta}$.*

From (15), we can see that $\bar{x}$ strictly decreases in β and converges to x^* as $\beta \to 1$ for any α bounded away from zero. Therefore, the investment strategy becomes efficient in the limit. Moreover, trade becomes efficient in the limit, as the buyer's per-period trading probability $F_e(\bar{x})$ is bounded away from zero in both equilibria. Overall, the social welfare converges to the first best in the limit.

Lemma 6. *In the two-sided offers case with observable investments, for any α that is bounded away from zero, as β converges to one, $F_e(x)$ converges in distribution to a point mass at x^*. In addition, the social welfare converges to the first best.*

Lemma 6 shows that even if the investments are observable, when the sellers have strictly positive bargaining power, the investments become efficient in the limit. To understand this result, note that a seller is the residual claimant of his or her investments when making the offer. As the time between two meetings shrinks to zero, a positive α implies that the seller has the chance to make the offer almost immediately after entry. Therefore he or she becomes the full residual claimant and invests efficiently, although he or she obtains only an α share of the total surplus from trade. Moreover, the social welfare also converges to the first best, as both the investments and trades are efficient.

To avoid any confusion, I should stress here that this limiting result does not imply any discontinuity around $\alpha = 0$. For this result to hold, we require α to be bounded away from zero.

6.2 Steady State Equilibrium

Let us turn to the steady state equilibrium with unobservable investments. For any x on the support of $F_e(x)$, trade always takes place if the seller is selected to make the offer, as the surplus from trade $y_0 - x - \beta\pi - \beta U(x)$ must be non-negative to ensure a positive search stage payoff $U(x)$.

We can follow the same logic as in the baseline model to verify that the support of $F_e(x)$ is $[x^*, \bar{x}]$ with a mass point at x^*, and that buyers play a mixed pricing strategy over $[r_S(x^*), r_S(\bar{x})]$.

In equilibrium, a seller's ex ante payoff v and a buyer's ex ante payoff π must be non-negative. Similar to the observable benchmark case, the equilibrium could take one of the two forms, depending on whether $\alpha(y_0 - \bar{x})$ is larger than $c(\bar{x})$. The detailed argument is as follows. When the support of $F(x)$ is also $[x^*, \bar{x}]$, using the indifference conditions to solve for v and π, we need only focus on a seller with a cost $\bar{x}$ and a buyer who offers $r_S(\bar{x})$. Combining their value functions,

$$U(\bar{x}) = \frac{\alpha(y_0 - \bar{x} - \beta\pi)}{1 + \alpha\beta - \beta} = c(\bar{x}) + v$$
$$\frac{1-\alpha\beta}{1-\alpha}\pi = y_0 - \bar{x} - \beta c(\bar{x}) - \beta v$$

We can solve v and π as follows:

$$v = \alpha(y_0 - \bar{x}) - c(\bar{x}) \tag{16}$$

$$\pi = (1-\alpha)(y_0 - \bar{x}) \tag{17}$$

As long as $\alpha(y_0 - \bar{x}) - c(\bar{x}) \geq 0$, both v and π are positive.

When $\alpha(y_0 - \bar{x}) < c(\bar{x})$, some fraction of incumbent sellers have cost x_0 and never trade. We can verify that the fraction $F(x_0)$ and equilibrium payoffs are the same as calculated in the observable benchmark.

To sum up, the equivalence in the equilibrium outcomes of observable and unobservable investments still holds in this extension.

6.3 Comparative Statics and the Limiting Case

As the investment strategy has support $[x^*, \bar{x}]$ and the social welfare is same as in the observable benchmark, the following proposition holds.

Proposition 7. *In the unique steady-state equilibrium with two-sided offers, the highest production cost $\bar{x}$ strictly decreases in β and α. For any α that is bounded away from zero, the investment strategy converges in distribution to a mass point at x^* and the equilibrium social welfare converges to the first best as $\beta \to 1$ or $\alpha \to 1$.*

Notice here, although in the baseline model the investment strategy also converges to the first best, the mechanisms behind the results are quite different.

7 Other Extensions and Robustness

7.1 Other Extensions

Investments are Observable with a Positive Probability. As the previous analysis reveals, the social welfare that could be generated from positive investments is completely dissipated by delays in trade. If in the search stage investments are observed with a positive probability q per period, then profitable trades can be conducted with no delay when investments are observable. This conjecture is indeed correct. Given any search friction, partial information yields strictly higher social welfare than no information (and full information).

To be more precise, seller entrants continue to use a mixed investment strategy. Its support is now $[\underline{x}, x_0]$, with $\underline{x} > x^*$ uniquely pinned down by

$$c'(\underline{x}) = \frac{-(1-q)}{1-q\beta}$$

We have two observations regarding the support of the investment strategy. First, the most efficient seller now underinvests. It is because when their investments become observable, they are fully extracted. Second, as x_0 is still on the support, the seller's ex ante payoff equals zero.

However, the buyer's payoff is strictly higher than that in the baseline model.

$$\pi = (y_0 - x_0)(1-q) + q\int_{\underline{x}}^{x_0} [y_0 - x - \beta c(x)]dF(x) > y_0 - x_0$$

As a result, the social welfare is strictly higher than that in the baseline model (and in the perfectly observable benchmark).

Buyers Costly Verify Sellers' Type. As a related extension, we examine how equilibrium outcomes change when a buyer can costly verify the opponent's production cost once they meet.

Assume that before making the offer, a buyer can pay a verification cost $A > 0$ to perfectly observe the seller's production cost. To make the extension non-trivial, assume that the cost A is not too large so that a buyer is willing to verify in some situations. Now the buyer's strategy also includes the probability of verification. Let us use $a \in [0, 1]$ to denote the probability of verification.

First, the net surplus from trade $y_0 - x - \beta U(x) - \beta\pi$ must be positive for any x on the support. Therefore, after paying the verification cost, a buyer offers the reserve price $x + \beta U(x)$. This is exactly why a buyer may be willing to pay the verification cost. Instead of paying a high price x_0 (or a lower price with a smaller trading probability), he or she can trade with a lower price. In addition, the probability of verification, a, must be strictly less than one. Otherwise, none of the seller entrants invests, which makes it optimal not to verify.

Following the same argument, the investment strategy and pricing strategy without verification must be mixed strategies. In each period, a seller receives his or her reserve price with a probability of a and receives a random price offer with a probability of $1 - a$. The type of the most efficient sellers on the support $\underline{x}$ is determined by

$$c'(\underline{x}) = \frac{-(1-a)}{1-a\beta}$$

For a positive a, $\underline{x}$ is higher than the socially efficient production cost x^* and is strictly increasing in a.

If a buyer decides not to verify, he or she must be indifferent between reserve prices. Like before, the indifference condition pins down the stationary cost distribution $F(x)$ with support $[\underline{x}, x_0]$, where $\underline{x}$ is to be determined.

If a buyer decides to verify, then his or her expected payoff is

$$\pi = y_0 - (\underline{x} + \beta c(\underline{x}))F(\underline{x}) - \int_{\underline{x}}^{x_0} (x + \beta c(x))dF(x) - A$$

Therefore, if the buyer is indifferent between verifying and not verifying, it must be true that

$$x_0 = (\underline{x} + \beta c(\underline{x}))F(\underline{x}) + \int_{\underline{x}}^{x_0} (x + \beta c(x))dF(x) + A \tag{18}$$

The right hand side of (18) strictly increases in $\underline{x}$ and is strictly larger than the left-hand side when $\underline{x} = x_0$. Therefore, (18) either 1) uniquely determines $\underline{x}$ or 2) shows that the buyer strictly prefers not to verify if the equation has no solution. In the first case, the mixed strategy parameter a can be uniquely solved from $c'(\underline{x})$. In the second case, the equilibrium outcomes are the same as that in the baseline model.

Most of the equilibrium properties are preserved after allowing for costly verification. For instance, the agents' ex ante payoffs and social welfare are the same as before. Moreover, we can show that for any cost A, buyers choose not to verify with a large enough discount factor and hence we return to the baseline model. We know that the stationary cost distribution becomes less "uncertain" as the search frictions vanish. Hence, the value of information is smaller.

Lemma 7. *In the steady state equilibrium with costly verification, the seller's ex ante payoff is zero, the buyer's ex ante payoff and social welfare is $y_0 - x_0$.*

Moreover, given any $A > 0$, there exists a $\hat{\beta}$ such that for any $\beta > \hat{\beta}$, buyers choose not to verify and the equilibrium outcomes are the same as that in the baseline model.

7.2 Robustness

The intuition for mixed strategy does not rely too much on the specific setup of the model. Moreover, as long as the agents continue to play the mixed strategy, most of the main results continue to hold. In this section, we check the robustness of the results of the baseline model against some alternative assumptions.

General One-to-One Matching Technology. The baseline model assumes that each player is definitely paired with one player from the other side in each period. The main results are robust if instead we have a general one-to-one matching technology so that the probability of not being paired in one period is positive.

To be more precise, the equilibrium investment strategy $F_e(x)$ remains non-degenerate with convex support $[\underline{x}, x_0]$ and a buyer plays the mixed pricing strategy over the reserve prices of these types. The difference is that $\underline{x}$ is higher than x^* if a seller cannot be paired with a probability of one in each period. Moreover, because the indifference conditions continue to hold, the equilibrium payoffs and social welfare equal the values generated with observable investments. Finally, as β converges to one, $\underline{x}$ converges to x^*. As meetings become more frequent, it is as if the most efficient sellers can trade immediately and hence they invest efficiently. The convergence results of $F(x)$ and $F_e(x)$ can also be extended with different rates of convergence that depend on the matching technology.

Ex ante Heterogeneous Sellers. We assume that all of the sellers are ex ante identical in the baseline model. The intuition can be extended to settings with ex ante heterogeneous sellers provided that their ex ante types remain unobservable and that their ex ante type distribution satisfies some mild assumptions.

As a simple example, suppose some fraction of the seller entrants are born with the production cost x^* and hence there is no need for them to invest. Others have the initial production cost x_0 and investment opportunity specified in the baseline model. We can easily verify that as long as the fraction of the efficient type is smaller than $F_e(x^*)$ (which is easier to be satisfied when β is large enough), all of the conclusions in the baseline model continue to hold.

Exogenous Death Shock. Suppose each player experiences an exogenous death shock with positive probability. For most of the analysis, this is equivalent to re-defining a smaller discount factor that also converges to one in the limit. The only complication is that now sellers also exit due to the death shock. Thus, we need to rewrite the stationary distribution condition accordingly. In other words, the investment strategy is slightly changed.

If we denote the discount rate as r_1 and the rate of the death shock as r_2, then the limit of $F_e(x^*)$ is continuous and decreasing in r_2, and is bounded away from zero for any r_2.

Lemma 8. *As the time between two consecutive meetings shrinks to zero,*

$$F_e(x^*) \to [1 + \frac{x_0 - x^* - c(x^*)}{y_0 - x_0} \frac{r_2}{r_1 + r_2}]^{-1}$$

Therefore, the limit of $F_e(x^*)$ decreases in the rate of death shock r_2 and is larger than $[1 + \frac{x_0 - x^* - c(x^*)}{y_0 - x_0}]^{-1}$.

8 Conclusion

This paper examines the investment incentive and its welfare consequences in an infinite horizon random search and bargaining game with unobservable and selfish investments.

We demonstrate that in a unique steady state equilibrium, both the investment strategy and the price offer distribution are non-degenerate with convex supports. Unobservability generates rent for high investment and therefore incentivizes investment even if the sellers have no bargaining power.

However, positive investments above the minimum level fail to generate any social welfare for any search friction. Trading inefficiency caused by unobservability erodes the welfare gain that could be created.

Moreover, we show that if the buyers have all of the bargaining power, then as meetings become more frequent the investment distributions of the incumbents and entrants shift in the opposite direction. The incumbent investment distribution converges to a point mass at no investment, and an entrant's investment becomes efficient.

Appendices

A Proofs for the Baseline Model

Appendix A.A Proof of Lemma 1

1. $U(x)$ **is Strictly Decreasing.** As a type x seller can always adopt the reserve price of a type $x+\epsilon$ seller (for some $\epsilon > 0$), $U(x)$ must be strictly decreasing in x.

2. $r_S(x)$ **is Strictly Increasing.** Multiply both sides of the seller's value function

by β and add x, we obtain the following equation after rearranging:

$$r_S(x) - \beta[E(p \mid p \geq r_S(x))(1 - H(r_S(x)) + Pr(p = r_S(x))) \\ + r_S(x)(H(r_S(x)) - Pr(p = r_S(x)))] = (1 - \beta)x \tag{19}$$

The left-hand side strictly increases in $r_S(x)$ and the right-hand side strictly increases in x. Therefore, $r_S(x)$ is strictly increasing in x.

3. $U(\bar{x}) = 0$. Because $r(\bar{x})$ is the highest price that a buyer is willing to offer, (1) with $x = \bar{x}$ becomes $U(\bar{x}) = \beta U(\bar{x})$. Hence, $U(\bar{x}) = 0$.

4. $U(x)$ **and** $r_S(x)$ **are Continuous.** No buyer offers a price higher than $r_S(\bar{x})$. Therefore, for any type $x > \bar{x}$, $H(r_S(x)) = 1$ and $U(x) = 0$.

   The previous step shows that $U(\bar{x}) = 0$.

   Because $U(x)$ decreases in x for any $x < \bar{x}$, $U(x)$ only has downward jumps. Suppose $U(x)$ jumps down at some point $\hat{x}$. Given that $r_S(x) = x + \beta U(x)$, $r_S(x)$ jumps downward at the same point $\hat{x}$. However, this contradicts $r_S(x)$ being strictly increasing.

   Therefore, both $U(x)$ and $r_S(x)$ are continuous.

Appendix A.B Proof of Proposition 1

1. Supports of $F(x)$, $F_e(x)$ and $H(p)$.

   The supports are closed given the assumption that the sellers and buyers trade when indifferent.

   Next, we show that the supports are convex. First, if a price offer p is on the support of $H(p)$, then $\hat{x}(p)$ must be on the support of $F(x)$ and $F_e(x)$. Otherwise, the buyer who offers price p is not optimizing because he or she can lower the price to $r_S(x')$ without affecting the probability of trade, where x' is the highest production cost among all of the costs on the support and lower than $\hat{x}(p)$.

Now suppose that there exist p_1, p_2 on the support of $H(p)$, such that any $p \in (p_1, p_2)$ is not on the support. As p_1 and p_2 are on the support, there exist worker types x_1 and x_2 on the support such that $r_S(x_i) = p_i$, $i = 1, 2$. For any $x \in (x_2, x_1)$, $U'(x)$ is a constant because

$$U'(x) = \frac{-1 + H(p_2)}{1 - \beta H(p_2)}$$

On the other hand, $c'(x)$ strictly increases in x. Together with the continuity of $U(x)$, it is impossible to satisfy the indifference condition $U(x_1) - c(x_1) = U(x_2) - c(x_2) \geq U(x) - c(x)$, for any $x \in (x_2, x_1)$. Therefore, the support of $H(p)$ is convex. By the continuity of $r_S(x)$, the support of $F(x)$ and $F_e(x)$ is also convex.

The lower bound of the price offers in equilibrium is never lower than the reserve price of the most efficient seller, i.e., $H(r_S(\underline{x})) - Pr(\tilde{p} = r_X(\underline{x})) = 0$. Plugging it into the envelope condition, $U'(\underline{x}) = c'(\underline{x}) = -1$. This implies that $\underline{x} = x^*$.

Hence, the support of $F(x)$ and $F_e(x)$ is $[x^*, x_0]$ and the support of $H(p)$ is $[r_S(x^*), r_S(x_0)]$.

2. $H(p)$ has no point mass.

   $U(x)$ is differentiable for any x on the support, because 1) the support is convex, 2) $U(x) - c(x) = 0$ and 3) $c(x)$ is differentiable. This implies that $r_S(x)$ and $\hat{x}(p)$ are differentiable for any x and p on the support. Hence, we can solve $H(p)$ from the equilibrium condition $U'(x) = c'(x)$ as follows:

$$H(r_S(x)) = \frac{1 + c'(x)}{1 + \beta c'(x)} \Rightarrow H(p) = \frac{1 + c'(\hat{x}(p))}{1 + \beta c'(\hat{x}(p))} \tag{20}$$

   It is straightforward to verify that $H(p)$ has no atom.

3. $F(x)$ and $F_e(x)$ have a unique point mass at x^*.

   We know that $r_S(x^*)$, which is only accepted by sellers with a cost of x^*, is on the support of $H(p)$. For a buyer who offers this price, he or she would get zero payoff if $F(x^*) = 0$. If this is the case, then the buyer would find it profitable to deviate to $r_S(x_0) = x_0$ which yields a positive payoff. Therefore, there must be a point mass at x^*.

Next we show that x^* is the unique point mass. Suppose these is another point mass at $x \in (x^*, x_0]$. Then there exist an ϵ, such that buyers would rather not offer price $p \in (r_S(x-\epsilon), r_S(x))$. By raising the price by a small amount, buyers can enjoy a discontinuous upward jump of the trading probability. This contradicts the convexity property of the support.

Appendix A.C Proof of Proposition 3

$F(x)$ has first order stochastic dominance over $F_e(x)$ requires that $F_e(x)-F(x) \geq 0$ for all x and that the inequality is strict for some x.

$$
\begin{aligned}
F_e(x) - F(x) &= \frac{F(x) - \int_{x^*}^{x} H(r_S(\tilde{x}))dF(\tilde{x})}{1 - \int_{x^*}^{x_0} H(r_S(\tilde{x}))dF(\tilde{x})} - F(x) \\
&= F(x)\frac{\int_{x^*}^{x_0} H(r_S(\tilde{x}))dF(\tilde{x}) - \int_{x^*}^{x} H(r_S(\tilde{x}))d\frac{F(\tilde{x})}{F(x)}}{1 - \int_{x^*}^{x_0} H(r_S(\tilde{x}))dF(\tilde{x})}
\end{aligned}
$$

Therefore, $F_e(x) - F(x) \geq 0$ for any $x \in [x^*, x_0]$ and the inequality is strict except for $x = x_0$.

Appendix A.D Proof of Theorem 2

1. We have shown the first two parts of the theorem.

2. By (5), the proportion of seller entrants who choose to invest efficiently is

$$
\begin{aligned}
F_e(x^*) &= \frac{F(x^*)}{1 - \int_{x^*}^{x_0} H(r_S(x))f(x)dx} \\
&= [1 + \frac{1-F(x^*)}{F(x^*)}\int_{x^*}^{x_0}(1 - H(r_S(x)))\frac{f(x)}{1-F(x^*)}dx]^{-1} \\
&= [1 + A]^{-1}
\end{aligned}
$$

where $A = \frac{1-F(x^*)}{F(x^*)} \int_{x^*}^{x_0} (1 - H(r_S(x))) \frac{f(x)}{1-F(x^*)} dx$.

By Median Value Theorem, there exist a $\tilde{x} \in (x^*, x_0)$ such that

$$\begin{aligned} A =& \frac{1 - F(x^*)}{F(x^*)} [1 - H(r_S(\tilde{x}))] \\ =& \frac{-c'(\tilde{x})(x_0 - x^* - \beta c(x^*))}{(y_0 - x_0)(1 + \beta c'(\tilde{x}))} \end{aligned}$$

Take derivative with respect to β

$$\frac{\partial A}{\partial \beta} = \frac{-c''(\tilde{x}) \frac{\partial \tilde{x}}{\partial \beta}(x_0 - x^* - \beta c(x^*)) + c'(\tilde{x})[c(x^*) + c'(\tilde{x})(x_0 - x^*)]}{(y_0 - x_0)(1 + \beta c'(\tilde{x}))^2}$$

Therefore, the sufficient conditions for the derivative to be negative are

$$\frac{\partial \tilde{x}}{\partial \beta} > 0 \text{ and } c(x^*) + c'(\tilde{x})(x_0 - x^*) > 0$$

We first show that the first condition is satisfied. $\tilde{x}$ is implicitly defined by

$$\int_{x^*}^{x_0} \frac{1 - H(r_S(x))}{1 - \beta} d \frac{F(x)}{1 - F(x^*)} = \frac{-c'(\tilde{x})}{1 + \beta c'(\tilde{x})} \tag{21}$$

As the preceding analysis shows, $\frac{1-H(r_S(x))}{1-\beta}$ strictly decreases in β for any $x \in (x^*, x_0)$. In addition, the mass of the conditional distribution $F(x \mid x > x^*)$ shifts to higher x's, where $\frac{1-H(r_S(x))}{1-\beta}$ is smaller. As a result, the left-hand side of (21) strictly decreases in β. At the same time,

$$\frac{\partial \frac{-c'(\tilde{x})}{1+\beta c'(\tilde{x})}}{\partial \beta} = \frac{-c''(\tilde{x}) \frac{\partial \tilde{x}}{\partial \beta} + [c'(\tilde{x})]^2}{(1 + \beta c'(\tilde{x}))^2}$$

Since the derivative has to be negative so that (21) holds, $\frac{\partial \tilde{x}}{\partial \beta}$ must be strictly positive.

This in turn implies that we only need to verify the second sufficient condition when $\beta = 0$, as $c'(\tilde{x})$ is the smallest when $\beta = 0$. It is straight forward to check that the density of the stationary distribution strictly increases in x, as

$$f(x) = \frac{(y_0 - x_0 - \beta\pi)(1 + \beta c'(x))}{[y_0 - \beta\pi - x - \beta c(x)]^2}$$

When $\beta = 0$, $1 - H(r_S(x)) = -c'(x)$. Therefore, equation (21) becomes,

$$\begin{aligned}
&- c'(\tilde{x}) = \int_{x^*}^{x_0} -c'(x)\frac{f(x)}{1 - F(x^*)}dx \\
&\text{where the right-hand-side } < \int_{x^*}^{x_0} -c'(x)\frac{1}{x_0 - x^*}dx = \frac{c(x^*)}{x_0 - x^*} \\
&\Rightarrow c(x^*) + c'(\tilde{x})(x_0 - x^*) > 0
\end{aligned}$$

We have proved that both of the sufficient conditions hold and thus

$$\frac{\partial F_e(x^*)}{\partial \beta} > 0$$

Next, we prove that $F_e(x^*) \to 1$ in the limit. $\tilde{x}$ is defined by

$$\int_{x^*}^{x_0} (1 - H(r_S(x)))\frac{f(x)}{1 - F(x^*)}dx = [1 - H(r_S(\tilde{x}))] \tag{22}$$

For any $\epsilon \in (0, x_0 - x^*)$, we can rewrite the left-hand side of (22) as

$$\begin{aligned}
&\int_{x^*}^{x_0-\epsilon} (1 - H(r_S(x)))\frac{f(x)}{1 - F(x^*)}dx + \int_{x_0-\epsilon}^{x_0} (1 - H(r_S(x)))\frac{f(x)}{1 - F(x^*)}dx \\
=&[1 - H(r_S(x_1))]\frac{F(x_0 - \epsilon) - F(x^*)}{1 - F(x^*)} + [1 - H(r_S(x_2))]\frac{1 - F(x_0 - \epsilon)}{1 - F(x^*)}
\end{aligned} \tag{23}$$

where $x_1 \in (x^*, x_0 - \epsilon)$ and $x_2 \in (x_0 - \epsilon, x_0)$.

Here $F(x_0 - \epsilon) - F(x^*)$ can be rewritten as,

$$(1 - \beta)(y_0 - x_0)\frac{-x^* - \beta c(x^*) + (x_0 - \epsilon) + \beta c(x_0 - \epsilon)}{(y_0 - \beta\pi - x_0 + \epsilon - \beta c(x_0 - \epsilon))(y_0 - \beta\pi - x^* - \beta c(x^*))}$$

We can verify that for any ϵ and ϕ, there exists an $\eta > 0$, such that when $1-\beta < \eta$, $F(x_0-\epsilon) - F(x^*) < \phi$. Given that $1-H(r_S(x))$ is bounded for any x, $\tilde{x}$ converges to x_2 in the limit. Because x_2 is in the interval $(x_0-\epsilon, x_0)$, for small enough ϵ, $\tilde{x}$ converges to x_0.

Therefore, A converges to zero in the limit. This in turn implies that $F_e(x^*) = [1+A]^{-1}$ converges to one.

Appendix A.E Proof of Proposition 4

In the first period, the measure of exits is e_1, which is also the proportion of incumbents who exit the market in the following periods. Denote the measure of incumbents at the beginning of period t by u_t. Then for any $t \geq 1$

$$u_{t+1} = 1 + u_t(1-e_t)$$

Multiply both sides by e_1 and rearrange, the above equation becomes

$$e_{t+1} = e_t + e_1(1-e_t)$$

First, we can verify that $e_t < 1$ for any t. When $t = 1$, e_1 is strictly less than 1 by construction. For any $t > 1$, the above condition shows that e_t is a convex combination of 1 and e_1 and hence is also strictly less than 1.

This equation also implies that $\{e_t\}$ is an increasing sequence, as $e_{t+1} - e_t = e_1(1-e_t) > 0$.

An increasing and bounded sequence must have a limit. Denote the limit as e_∞, which can be solved from

$$e_\infty = e_\infty + e_1(1-e_\infty) \Rightarrow e_\infty = 1.$$

The rest of the proposition is already proved in the main body of this paper.

B Proofs for the Two-Sided Investment Case

Appendix B.A Proof of Lemma 2

1. $p(y)$ **is increasing**.

   If the support of $G(y)$ is degenerate, then we have nothing to prove.

   Otherwise, consider any y_1 and y_2 on the support with $y_1 > y_2$. Denote p_i as $p(y_i)$ for $i = 1, 2$. Because p_1 (p_2) solves the maximization problem of a type y_1 (y_2) buyer, the following two inequalities must hold

   $$(y_1 - p_1)F(\hat{x}(p_1)) + [1 - F(\hat{x}(p_1))]\beta\Pi(y_1) \geq (y_1 - p_2)F(\hat{x}(p_2)) + [1 - F(\hat{x}(p_2))]\beta\Pi(y_1)$$
   $$(y_2 - p_2)F(\hat{x}(p_2)) + [1 - F(\hat{x}(p_2))]\beta\Pi(y_2) \geq (y_2 - p_1)F(\hat{x}(p_1)) + [1 - F(\hat{x}(p_1))]\beta\Pi(y_2)$$

   Adding the two equations we have the following inequality

   $$(y_1 - y_2)[F(\hat{x}(p_1)) - F(\hat{x}(p_2))] \geq [F(\hat{x}(p_1)) - F(\hat{x}(p_2))][\beta\Pi(y_1) - \beta\Pi(y_2)]$$

   In equilibrium, $\Pi(y_1) - \Pi(y_2) = e(y_1) - e(y_2)$, which implies that $y_1 - y_2 > \beta[\Pi(y_1) - \Pi(y_2)]$. Given this, the above inequality then shows that $F(\hat{x}(p_1)) \geq F(\hat{x}(p_2))$. This proves that $p(y)$ increases in y.

2. $P(y)$ **is single-valued**.

   We can prove that the support of $F(x)$ is $[x^*, x_0]$ using the previous approach. Therefore, $F(x)$ is a strictly increasing function of x for any $x \in [x^*, x_0]$.

   Consider any y on the support of $G_e(y)$ and denote the corresponding optimal price as p. The search stage payoff is

   $$\Pi(y) = \max_p \frac{[y - p]F(\hat{x}(p))}{1 - \beta\,[1 - F(\hat{x}(p))]}$$

   which is strictly convex in p. Therefore, $P(y)$ is single-valued.

Appendix B.B Proof of Proposition 5

1. We can apply the same proof in the baseline model to show that the support of $F_e(x)$ and $F(x)$ is $[x^*, x_0]$ and that x^* is the unique point mass.

2. The compactness of the support of $G_e(y)$ and $G(y)$ results from the fact that y^* is finite and that all of the agents choose to trade when feeling indifferent.

   To see the convexity property, suppose that y_1, y_2 are on the support and any $y \in (y_1, y_2)$ is not. Then it must be the case that $p(y_1) < p(y_2)$. Otherwise $\Pi'(y)$ is constant in the interval and the indifference condition cannot be satisfied. Combined with the monotonicity of $p(y)$, the fact that $y \in (y_1, y_2)$ are on the support implies that $p \in (p(y_1), p(y_2))$ are not on the support of $H(p)$. This leads to a contradiction.

   Because the support of $G(y)$ is convex, we can use the envelope and indifference condition to determine $\bar{y}$ and $\underline{y}$.

$$e'(\bar{y}) = \Pi'(\bar{y}) = 1 \text{ and } e'(\underline{y}) = \Pi'(\underline{y}) = \frac{F(x^*)}{1 - \beta(1 - F(x^*))}$$

   As a result, $\bar{y} = y^*$ and $\pi = y^* - x_0 - e(y^*)$. Plug π into the condition for $\underline{y}$, it becomes

$$y^* - x_0 - e(y^*) = (\underline{y} - x^* - \beta c(x^*))e'(\underline{y}) - e(\underline{y})$$

   $\underline{y}$ is uniquely determined by the above equation and $\underline{y} < y^*$. To see this, notice that the right-hand side of the above equation strictly increases in $\underline{y}$. It equals zero when $\underline{y} = y_0$ and it is strictly larger than the left-hand side when $\underline{y} = y^*$. Therefore, two sides of the equation can cross each other only once and the intersection point is strictly smaller than y^*.

   Finally, no point mass on the support of $H(p)$ implies that the same property holds for $G(y)$.

Appendix B.C Proof of Lemma 3

Combining the envelope condition for $\Pi(y)$ and the indifference condition that $\Pi'(y) = e'(y)$ for any y, we have

$$F(\hat{x}(p(y))) = \frac{(1-\beta)e'(y)}{1-\beta e'(y)} \text{ for any } y \text{ on the support}$$

We can use the indifference condition $\Pi(y) - e(y) = \pi$ to solve $p(y)$ as follows:

$$\Pi(y) - e(y) = (y - p(y))e'(y) - e(y) = y^* - x_0 - e(y^*)$$
$$\Rightarrow p(y) = y - \frac{e(y) + y^* - x_0 - e(y^*)}{e'(y)}$$

$p(y)$ is continuous and strictly increasing in y. Therefore, the inverse function $y(p)$ is well defined. We can then plug it in to solve $F(x)$ and $G(y)$.

From the preceding discussion, we can see that $F(x)$, $F_e(x)$, $G(y)$ and $G_e(y)$ as defined are the only distributions that satisfy all equilibrium conditions. Hence, the steady state equilibrium is unique.

Appendix B.D Proof of Proposition 6

1. By (10), the implicit function theorem tells us that

$$[\underline{y} - x^* - \beta c(x^*)]e''(\underline{y})\frac{\partial \underline{y}}{\partial \beta} = c(x^*)e'(\underline{y})$$

Therefore, $\frac{\partial \underline{y}}{\partial \beta}$ is strictly positive.

Denote the $t * 100^{th}$ percentile of $G(y)$ with β as $y_{t,\beta}$, i.e.,

$$G(y_{t,\beta}) = \frac{1 + c'(\hat{x}(p(y_{t,\beta})))}{1 + \beta c'(\hat{x}(p(y_{t,\beta})))} = t$$

Rearrange the above equation

$$1 - t = -(1 - t\beta)c'(\hat{x}(p(y_{t,\beta})))$$

Therefore, when β increases, $\hat{x}(p(y_{t,\beta}))$ strictly decreases.

2. $\lim_{\beta\to 1} \underline{y} < y^*$ can be shown by plugging $\beta = 1$ and $\underline{y} = y^*$ into equation (10), and check that the left-hand side is strictly smaller the right-hand side.

   For any $y > \underline{y}$, $\hat{x}(p(y)) > x^*$. Therefore, the denominator of $G(y)$, $1 + c'(\hat{x}(p(y)))$, is strictly positive. Then it is straightforward to verify that as $\beta \to 1$, $G(y) \to 1$ for any $y > \underline{y}$.

   For any $y > \underline{y}$, there exist $\check{y}$ and $\check{\check{y}}$, such that

$$\begin{aligned} G_e(y) &= \frac{\int_{\underline{y}}^{y} F(\hat{x}(p(\tilde{y})))dG(\tilde{y})}{\int_{\underline{y}}^{y^*} F(\hat{x}(p(\tilde{y})))dG(\tilde{y})} = \frac{F(\hat{x}(p(\check{y})))G(y)}{F(\hat{x}(p(\check{\check{y}})))} \\ &= \frac{e'(\check{y})(1-\beta e'(\check{\check{y}}))}{e'(\check{\check{y}})(1-\beta e'(\check{y}))} G(y) \end{aligned}$$

   When $\beta \to 1$, both $\check{y}$ and $\check{\check{y}}$ approaches $\underline{y}$, following the same argument in the proof for proposition 2, and $G(y)$ approaches one for any $y > \underline{y}$. Therefore, $G_e(y) \to 1$ for any $y > \underline{y}$.

C Proofs for the Two-Sided Offer Case

Appendix C.A Proof of Lemma 4

Suppose there are more than one optimal investment levels in a stationary equilibrium. Given condition (15), there can be at most one x with non-negative net surplus.

This means that if there exists another production cost of $\hat{x}$ on the support, the associated surplus from trade must be negative. As a result, a seller of cost $\hat{x}$ can never trade and his or her ex ante payoff is $v = -c(\hat{x})$. So the only possibility is that $\hat{x} = x_0$.

The sellers with the production cost x_0 never leave the market. Therefore, the entrant's type can only be the invested one in a stationary equilibrium.

Appendix C.B Proof of Lemma 5

Equilibrium 1.

Assume $\alpha(y_0 - \bar{x}) \geq c(\bar{x})$.

All seller entrants invest to decrease their production costs to $\bar{x}$. Agents' ex ante payoffs can be solved as $v = \alpha(y_0 - \bar{x}) - c(\bar{x})$ and $\pi = (1-\alpha)(y_0 - \bar{x})$. By the assumption, v is positive and all seller entrants invest positive amount.

Equilibrium 2.

Assume $\alpha(y_0 - \bar{x}) \leq c(\bar{x})$.

If a seller entrant invests, he or she chooses to decrease the production cost to $\bar{x}$. Given π, his or her ex ante payoff $v = \frac{\alpha}{1+\alpha\beta-\beta}(y_0 - \bar{x} - \beta\pi) - c(\bar{x})$. To make sure that sellers are indifferent between $\bar{x}$ and x_0, v must equal zero. As a consequence,

$$\pi = \frac{\alpha(y_0 - \bar{x}) - (1 + \alpha\beta - \beta)c(\bar{x})}{\alpha\beta}$$

Denote $F(\bar{x})$ as q, we can also solve π from the buyer's value function as follows:

$$\pi = \alpha\beta\pi + (1-\alpha)[q(y_0 - \bar{x} - \beta c(\bar{x})) + (1-q)\beta\pi]$$

Equating two π's, q can be solved as:

$$q = \frac{(1-\beta)\pi}{(1-\alpha)(y_0 - \bar{x} - \beta c(\bar{x}) - \beta\pi)}$$

As q is a probability, it lies between zero and one. q is always positive by the above equation. The requirement that q is less than one is equivalent to $\alpha(y_0 - \bar{x}) \leq c(\bar{x})$.

The last equilibrium condition we need to verify is $y_0 - x_0 - \beta\pi \leq 0$. After plugging in the expression of π, this condition becomes $c(\bar{x}) \leq \frac{\alpha}{1+\alpha\beta-\beta}(x_0 - \bar{x})$. This inequality holds since $c'(\bar{x}) = \frac{-\alpha}{1+\alpha\beta-\beta}$ and $c(x)$ is strictly convex.

Appendix C.C Proof of Lemma 6

As $\beta \to 1$, because α is bounded above zero, the right-hand side of (15) converges to -1, which implies $\bar{x} \to x^*$. Because $F_e(x)$ is a point mass at $\bar{x}$ in both types of equilibrium, it converges to a point mass at x^*.

In equilibrium 1, the social welfare equals $y_0 - \bar{x} - c(\bar{x})$. Hence, it converges to the first best. In equilibrium 2, the social welfare equals π, which also converges to $y_0 - x^* - c(x^*)$, given that α is bounded above zero.

D Proofs for Robustness and Other Extensions

Appendix D.A Proof of Lemma 7

x_0 is on the support. Hence, the first part of the proposition holds.

As $\beta \to 1$, $F(x) \to 0$ for any $x < x_0$. Therefore, the right-hand side of (18) converges to $x_0 + A$, which is strictly larger than the left-hand side. This means that a buyer does not pay to verify if β is close enough to 1.

Moreover, the right-hand side shifts up with a larger β. It follows from 1) $F(x)$ increases in β, which is equivalent to putting higher weights on larger reserve prices and 2) the reserve price $x + \beta c(x)$ increases in β for any x.

Given the above two facts, there exists a $\hat{\beta}$ for any A, such that the buyers choose not to verify when $\beta > \hat{\beta}$.

Appendix D.B Proof of Lemma 8

Denote the probability of surviving a death shock as $\delta = e^{-r_2 t}$, where t is the length of the time between two periods. Then

$$\begin{aligned} F_e(x^*) =& \frac{F(x^*)}{1 - \delta \int_{x^*}^{x_0} H(r_S(x)) f(x) dx} \\ =& [1 + \frac{1 - F(x^*)}{F(x^*)} \int_{x^*}^{x_0} [1 - \delta H(r_S(x))] \frac{f(x)}{1 - F(x^*)} dx]^{-1} \\ =& [1 + B]^{-1} \end{aligned}$$

Plug in $F(x^*)$ and $H(r_S(x))$, and use the Median Value Theorem, there exists a $\tilde{x} \in (x^*, x_0)$ such that

$$B = \frac{x_0 - x^* - \hat{\beta}c(x^*)}{(1-\hat{\beta})(y_0 - x_0)} \frac{1 - \delta + (\hat{\beta} - \delta)c'(\tilde{x})}{1 + \hat{\beta}c'(\tilde{x})}$$

Here $\hat{\beta} = \beta\delta$. Take $t \to 0$,

$$\lim_{t\to 0} B = \frac{x_0 - x^* - c(x^*)}{(y_0 - x_0)} \lim_{t\to 0} \left[\frac{1}{1 + e^{-(r_2+r_1)t}c'(\tilde{x}_t)} + \frac{1 + c'(\tilde{x}_t)}{1 + e^{-(r_2+r_1)t}c'(\tilde{x}_t)} \frac{e^{-r_1 t} - 1}{1 - e^{-(r_2+r_1)t}} \right]$$

By the same argument in the proof for Theorem 2, the limit of $\tilde{x}_t$ is x_0. Using the L'Hopital's rule

$$\lim_{t\to 0} B = \frac{x_0 - x^* - c(x^*)}{y_0 - x_0} \frac{r_2}{r_1 + r_2}$$

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