Search with Private Information: Sorting, Price Formation, and Convergence to Perfect Competition

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December 2016

Abstract

We consider a dynamic market where heterogeneous buyers and sellers are randomly paired in each period, with match values being supermodular (submodular) in both types. Within each match, the seller type become observable while the buyer type remains private information, and the seller makes a take-it-or-leave-it offer. We first establish the existence of steady state equilibrium and then characterize properties of sorting as well as price offers given two extreme search frictions. When agents completely disregard for future payoffs, a stronger condition of log-supermodularity (log-submodularity) of the match value is necessary and sufficient for positive (negative) assortative matching. When search frictions vanish, the condition for positive (negative) sorting returns to supermodularity (submodularity) and the price offers are such that agents receive their marginal contributions to the match. In this setting with supermodular or submodular valuations, we demonstrate that as search frictions diminish to zero, search equilibria converge to perfect competition.

Keywords: Private Information, Sorting, Price, Convergence

JEL classifications: C78, D41, D83

1. Introduction

The value of a traded good often depends on both buyer and seller types. When there is complementarity between the two sides, investigating the sorting pattern is crucial for studying the outcomes of decentralized trades. To this end, this paper focuses on large markets with information and search frictions and looks into the underlying environmental factors that influence the agents' sorting decisions, the speed of trade, and the division of the surplus. Moreover, we would like to ask whether the trading outcomes become socially efficient as search frictions diminish. To the best of our knowledge, this is the first attempt in the literature to show the convergence in a setting with supermodular (or submodular) interdependent valuations and private information.

Becker (1973) serves as a benchmark characterization of sorting—in a "frictionless" world, Positive Assortative Matching (henceforth PAM) arises when the match value is supermodular in types, while Negative Assortative Matching (NAM) occurs with submodular match values. We depart from Becker's setting by restricting the interactions of agents in two ways. First, it is difficult to meet potential trading partners, as each buyer encounters only a random seller in each period (and vice versa). Second, in each meeting of potential trading partners, the seller cannot observe the buyer's type.

More precisely, our environment is a repeated, bilateral matching market with exogenous entry. Buyers and sellers are heterogeneous and have persistent types. They are randomly matched pairwise in each period. In each match, the seller's type can be jointly observed, but only the buyer knows her own type (and thus knows the match value). In turn, the seller has all of the bargaining power and can make a take-it-or-leave-it price offer. If the pair agree to trade, the buyer pays the price and receives the joint output (we will use "output" and "match value" interchangeably throughout the paper), which is an increasing function of both agents' types, and both of them leave the market permanently. Otherwise, the pair is dissolved and they are randomly matched with some other agents in the next period if they survive an exogenous exit shock. The exit shock captures the search frictions in this model.

This setting reflects a broad set of applications—it is straightforward to imagine a market for vertically-differentiated products, where consumers differ in preferences across goods. In the housing market, for instance, homes for sale vary in kitchen appliances, and houses (sellers) are visited by buyers with different marginal returns to better appliances, depending on their frequencies/habits of cooking. Our model can be seen as the case in which the buyer's preference is her private information. Similarly, this might represent the hiring process in a

labor market where heterogeneous workers and firms produce joint output. For consistency, we describe the model's agents as buyers and sellers throughout the analysis to follow, but where appropriate, it will be insightful to draw motivation from other such applications.

Characterizing sorting seems very difficult initially, due to the fact that match values are interdependent and that an agent's expected payoff from searching depends on all of the other agents' strategies as well as the endogenous type distributions. But we are able to characterize properties of sorting by focusing on two extreme cases: when agents completely disregard for future payoffs and when search frictions approach zero. The former case reveals the economic forces that lead to sorting in our environment, which helps us understand the main result of the paper: as search frictions vanish, search equilibria converge to the competitive limit if the output function is supermodular (or submodular).

In the model, a seller chooses an optimal price. Equivalently, she is picking the type of the lowest marginal buyer who is indifferent between buying and not buying.[1] If the seller chooses a higher marginal type who values the product more, she obtains a higher price but the per-period trading probability is lower. In other words, information friction generates a trade-off for the seller between the terms of trade and probability of trade. Both the terms of trade and probability of trade and hence the trade-off depend on the seller's type. If the output function is supermodular, a seller of a higher type enjoys a larger increase in price by raising the marginal type, which leads to PAM, but also has more to lose from a delay in trade, which leads to NAM. As a result, supermodularity is generally not sufficient to ensure PAM. In addition, this suggests an interplay between information friction and search frictions: sellers care more about the probability of trade, which results from the information friction when search frictions are large.

Unfortunately, the simplicity of this intuition masks the underlying complexity of equilibrium decisions and interactions. In contrast to related models using Nash bargaining (e.g., Shimer and Smith (2000), Smith (2006) and Atakan (2006)), both prices *and* matching sets arise endogenously in our model. While we prove the existence of search equilibrium for any arbitrary search friction, we focus on two cases with the maximum and almost zero search frictions. We can neatly characterize the equilibrium in these two cases. The comparison between the two cases well illustrates the interplay between information and search frictions.

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[1]When trade is hindered by search friction, a seller in general could trade with buyers whose types lie in several disconnected intervals, as shown in Shimer and Smith (2000). In other words, a seller can have multiple marginal buyer types. However, once one marginal type is chosen, all the others are uniquely determined in equilibrium, as the seller only has one degree of freedom, which is the price. As a result, for the intuition presented here, we assume that the seller chooses the lowest marginal type without loss of generality.

In the static case in which all agents experience exit shock after one period, our analysis is simplified because the discounted continuation value is zero. We find that the direction of sorting depends on the log-supermodularity of output—log-supermodular production functions give rise to PAM, while log-submodularity leads to NAM. This is a stronger condition than the supermodularity that governs sorting in a frictionless market. The preceding discussions reveal that the incentive to secure trade pushes equilibrium sorting towards negative assortative even if the production function is supermodular. Therefore, an output function with stronger complementarity is required.

The tendency of NAM identified in the static case helps us understand why PAM is possible with a supermodular output function in the other extreme where the search frictions vanish. The diminishing search friction is modelled as the time between two successive periods shrinking to zero. Given that the rate of exit shock in one unit of time is constant, the expected number of periods before forced exit grows to infinity. As a result, the incentive to secure trade in any given period becomes inconsequential and hence supermodularity (submodularity) reemerges as the condition for the convergence to one-to-one PAM (NAM). This frictionless limit is particularly informative regarding sorting and how it depends on the interplay between private information and search frictions. Our analysis suggests generally that, when buyers have private information, sellers are less likely to capitalize on productive complementarities by sorting positively. This resistance to sorting disappears, however, when we reduce search frictions. In this sense, reducing search friction can help agents sort efficiently even in the presence of private information.

Another key result of this paper is the convergence to perfect competition, which requires the time to trade for any agent to converge to zero and the sorting pattern to converge to Becker's result. As the length of each period shrinks to zero, a seller becomes more picky and chooses a price that is only accepted by an arbitrarily small set of buyers with types neighboring a "targeted type". This "targeted type" is strictly increasing (decreasing) in the seller's type, as supermodularity (submodularity) means that a seller with a higher (lower) type is less costly in providing a certain level of utility to a buyer. This implies that the per-period trading probabilities for all agents diminish to zero. The expected time that it takes to trade therefore does not necessarily reduce to zero even if the trading opportunities appear arbitrarily quickly. In equilibrium, however, we can show that the per-period trading probability converges to zero at a smaller rate and hence any agent indeed expects to trade immediately after entry. This also means that the limiting sorting pattern coincides with Becker's result, as we have already shown that it is one-to-one positive (negative) assortative

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with a supermodular (submodular) output function.

In addition, we can characterize the equilibrium prices analytically in the limit. The price offers in the limit are such that agents appropriate their marginal contributions, which result from the weakened monopolistic aspect of bilateral trade as the time preference of agents is removed.

The paper proceeds as follows: Section 2 frames our findings in the context of related literature. Section 3 introduces the model. Section 4 lays out the equilibrium conditions, establishes the existence of a search equilibrium and formally defines sorting. Section 5 characterizes the sorting and price formation when the model is effectively static. Section 6 offers results for the other extreme case (as search frictions vanish) and shows that the search equilibria converge to perfect competition. Finally, section 7 draws connections between these limiting cases and concludes our paper.

2. Related Literature

Our analysis builds upon previous theoretical works related to assortative matching. Studies in this area often attempt to answer two primary questions: (1) What types of agents will match with each other? (2) What types of agents would match with each other to optimize overall welfare, and if these matching patterns differ, why? To properly place our analysis in the context of this literature, we will compare our setting to several that have been previously studied, both through objective sorting characterizations and through insights regarding the relationship between sorting, search, and asymmetric information.

A standard benchmark used for comparison is the frictionless "Walrasian" setting studied by Becker (1973) and Rosen (1974). In this environment, there is full information regarding prices and types, and meeting trading partners is fully costless for buyers and sellers. Becker famously demonstrated that supermodular production functions give rise to PAM in this environment.

Beyond this, though, recent studies have taken a renewed interest in sorting, trying to understand how it is impacted by departures from the frictionless benchmark. Among these frictional extensions, the setting we study is especially well-suited for comparison to those involving two particular classes of frictions—the bilateral monopoly which arises in random search and the coordination frictions in directed search. We elaborate upon these connections below.

There are obvious connections between our study and a series of papers on sorting with

random search and transferable utility (e.g., Shimer and Smith (2000) and Atakan (2006)). Our study differs from these primarily in its incorporation of buyer private information—in the studies mentioned above, there is full information and an exogenously given sharing rule in each meeting of potential partners. In terms of the main focuses, Shimer and Smith (2000) demonstrate that, due to search frictions, PAM requires more stringent conditions than supermodularity comparing to the frictionless benchmark. It provides sufficient conditions for sorting given some arbitrary search friction level. In this paper, we focus on how the conditions for sorting vary for different magnitudes of search frictions. We identify the source of resistance to sorting as information frictions in the static case, and extend the intuition to explain why the condition for sorting is weakened as search frictions diminish. In addition, compared to Shimer and Smith (2000), we are also particularly interested in investigating if and when the search equilibria converge to perfect competition.

There are also papers establishing relationships between coordination frictions and sorting patterns. Eeckhout and Kircher (2010) study a one-shot setting with buyer private information and directed search. They investigate the impact of coordination frictions on sorting, governed by the search technology, i.e., a function which maps the buyer-seller composition at each location to a realized number/measure of bilateral matches. They show that stronger complementarity is required when it is harder for a seller to successfully meet a buyer given a buyer-seller ratio. Note that, when viewed through the lens of timing, the two settings appear to be much more closely related. Eeckhout and Kircher's sellers use posted prices to sort agents prior to meeting (*ex ante* sorting), while our sellers sort only after the buyer has arrived (*ex post* sorting).

Our convergence result is also related to the literature on dynamic matching and bargaining. A large number of papers of matching and bargaining have investigated whether market equilibria converge to the competitive limit as search frictions vanish. The convergence result holds in settings with perfect information (e.g., Rubinstein and Wolinsky (1985) (1990), Gale (1986) (1987), Mortensen and Wright (2002)) and settings with two-sided private information (e.g., Satterthwaite and Shenyerov (2007) (2008), Atakan (2009), Shneyerov and Wong (2010), Lauermann (2013)). The main difference between this model and the previous works is the interdependence of the valuations. In previous works, the utility of consuming the product solely hinges on the buyer's type and hence a buyer's purchasing decision only depends on the price. In this model, a buyer also directly cares about the seller's type. This is why the previous convergence results are not applicable to the current setting. We show that efficiency is obtained in the limit with the presence of private information along with

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the return to supermodularity (submodularity) as sufficient for PAM (NAM). To the best of our knowledge, this is the first paper that shows the convergence to perfect competition with supermodular (submodular) interdependent valuations and private information.

3. The Model

We consider a discrete-time, dynamic model with heterogeneous buyers and sellers and focus on steady state equilibrium throughout the paper. There are equal measures of buyers and sellers in each period in the equilibrium of the steady state economy, where buyer types x are distributed according to cdf $F_B(x)$, and seller types y are distributed according to cdf $F_S(y)$.[2] Let $f_B(x)$ and $f_S(y)$ denote the corresponding steady state pdfs for buyer and seller types, respectively.

Each buyer is randomly matched with one seller (and vice versa) in each period. In each pair, the seller type becomes observable, while the buyer type remains private. Sellers, however, have the power to make take-it-or-leave-it offers $P(y)$ to buyers.[3] Therefore, the strategy of a type y seller is the price offer $P(y)$ and the strategy of a type x buyer is the set of sellers with whom he is willing to trade.

If trade occurs, output $z(x,y)$ is produced and both parties leave the market permanently with utility $z(x,y) - P(y)$ for the buyer and $P(y)$ for the seller. Those who do not trade experience an exogenous exit shock with probability $1 - \beta$, in which case they leave the market. Otherwise, remaining buyers and sellers play the same game in the following period, along with a continuum of newly entering buyers and sellers with measure 1. These new entrants are randomly drawn from fixed distributions with pdfs $\gamma_B(x)$ for buyers and $\gamma_S(y)$ for sellers over the bounded intervals $X = [\underline{x}, \overline{x}]$ and $Y = [\underline{y}, \overline{y}]$, assuming $\underline{x} > 0$ and $\underline{y} > 0$.

For convenience throughout the analysis, let μ denote a probability measure on the ranges of buyer and seller types $X \uplus Y$, where $\uplus$ is the disjoint union of X and Y (i.e., for a given set $\alpha \in X$, we can write $\mu(\alpha) = \Pr(\widetilde{x} \in \alpha \mid \widetilde{x} \in X) = \int_{\widetilde{x} \in \alpha} dF_B(\widetilde{x})$). Conceptually, we can think of μ as nesting within it two probability measures—one for buyers (μ_B) and one for sellers (μ_S). For notational simplicity, we omit these subscripts throughout, as the relevant population is generally obvious in each case.

Agents discount future payoffs only due to the possibility of leaving the market without

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[2]As will soon be obvious, these distributions have bounded supports in equilibrium.

[3]By Riley and Zeckhauser (1983), when a seller can commit to a selling mechanism when facing a buyer with an unknown type, the optimal mechanism is equivalent to a take-it-or-leave-it offer. Therefore, it is without loss of generality to assume that sellers make such price offers.

trading, so the relevant discount factor for all players is β.

For subsequent analysis, we impose the following assumptions on $z(x, y)$, $\gamma_B(x)$ and $\gamma_S(y)$:

Assumption 1: *Over the domain $X \times Y$, the output function $z(x, y)$ is*

1. *nonnegative and bounded above;*

2. *twice continuously differentiable;*

3. *strictly increasing in both arguments with uniformly bounded first partial derivatives; and*

4. *log-concave in x.*

Assumption 2: $\gamma_B(x)$ *is continuous over the domain of X and $\gamma_S(y)$ is continuous over the domain of Y. $\gamma_B(x)$ and $\gamma_S(y)$ are strictly positive and bounded above for any x and y.*

4. Steady State Equilibrium

Buyer's Problem

Given $\{P(y)\}_{y \in Y}$, a buyer with type x chooses a set of sellers with whom he is willing to trade. As shown in later sections, unlike in a frictionless market, equilibria do not entail a deterministic, one-to-one matching. Rather, the buyer trades probabilistically with a seller whose type is randomly drawn from a range of "acceptable" types. If he trades with a type y seller, the payoff is $z(x, y) - P(y)$. Otherwise, he expects to get some payoff in the next period, denoted as $V_B(x)$, which is discounted with factor β because of the exogenous exit shock. As a result, a type x buyer is willing to trade with a type y seller if $z(x, y) - P(y) \geq \beta V_B(x)$.

We therefore denote a type x buyer's surplus from trading with a type y seller as $s(x, y) = z(x, y) - \beta V_B(x) - P(y)$. If a seller's price $P(y)$ is acceptable by a type x buyer, then we say that y is in buyer x's *matching set*. Denote the equilibrium matching set as $\mathbb{M}_B(x)$, which is defined as

$$\mathbb{M}_B(x) = \{y : s(x, y) \geq 0\}$$

A buyer of type x trades if he meets a seller from his matching set. His value function can therefore be expressed as

$$V_B(x) = \int_{\mathbb{M}_B(x)} (z(x, y) - P(y)) dF_S(y) + \left(1 - \int_{\mathbb{M}_B(x)} dF_S(y)\right) \beta V_B(x)$$

Rearrange and we obtain

$$V_B(x) = \frac{\int_{\mathbb{M}_B(x)} (z(x,y) - P(y)) dF_S(y)}{1 - \beta + \beta \int_{\mathbb{M}_B(x)} dF_S(y)}$$

Seller's Problem

In turn, $\mathbb{M}_S(y)$ corresponds to the *matching set* of a type y seller, which is defined as

$$\mathbb{M}_S(y) = \{x : s(x,y) \geq 0\}$$

Obviously, $y \in \mathbb{M}_B(x)$ if and only if $x \in \mathbb{M}_S(y)$.

By choosing different prices, a type y seller changes the matching set, as the surplus $s(x,y) = z(x,y) - \beta V_B(x) - P$ strictly decreases in P. Therefore, given $V_B(x)$, a type y seller's matching set with a higher price is a subset of that which has a lower price, implying a lower probability of trade in one period. In other words, a seller faces the trade-off between the term of trade and the probability of trade.

Denote the value function of a seller of type y as $\Pi(y)$. Given $F_B(x)$ and $V_B(x)$, $\Pi(y)$ can be expressed as

$$\Pi(y) = \max_P \{ P \int_{\mathbb{M}_S(y;P,V_B(x))} dF_B(x) + (1 - \int_{\mathbb{M}_S(y;P,V_B(x))} dF_B(x)) \beta \Pi(y) \}$$

Steady State Condition

The last equilibrium condition is the steady state condition: the measure of outflow of any type must equal the measure of inflow of the same type. The inflow is governed by the entrant type distributions γ_B and γ_S. The outflow of buyers of type x consists of two groups. A buyer would exit if he is paired with a seller in his matching set. Otherwise, a buyer would leave the market because of the exit shock. The same accounting condition applies to the seller side.

The pdfs of the steady state type distributions (f_B, f_S) and pdfs of the entrant type

distributions (γ_B, γ_S) therefore must satisfy the two inflow-outflow equations below:

$$\hat{f}_B(x) = \frac{\gamma_B(x)}{(1-\beta) + \beta\frac{\int_{\mathbb{M}_B(x)} \hat{f}_S(y)dy}{\int \hat{f}_S(y)dy}}$$

$$\hat{f}_S(y) = \frac{\gamma_S(y)}{(1-\beta) + \beta\frac{\int_{\mathbb{M}_S(y)} \hat{f}_B(x)dx}{\int \hat{f}_B(x)dx}}$$

$$\text{where } f_B(x) = \frac{\hat{f}_B(x)}{\int \hat{f}_B(\tilde{x})d\tilde{x}} \text{ and } f_S(y) = \frac{\hat{f}_S(y)}{\int \hat{f}_S(\tilde{y})d\tilde{y}}$$

Here $\hat{f}_B(x)$ $(\hat{f}_S(y))$ stands for the measure of buyers (sellers) who have the type x (y) in the steady state.

We devote the remainder of this section to some technical preliminaries for the sorting analysis. First, we offer a fairly general existence proof for an equilibrium in which prices are continuous in seller type. Then we formally define PAM and NAM with non-degenerate matching set, following Shimer and Smith (2000).

4.1 Existence

We focus on equilibria where the price is continuous in seller type. For this reason, our existence proof initially assumes the continuity of $P(y, V_B(\cdot))$—the mapping that determines the optimal price given a particular seller type and a function specifying the outside option for each buyer type. We then verify at the end of the proof that the equilibrium price is indeed continuous.

We use the Schauder fixed point theorem in proving the existence. More precisely, we show that the mapping from the continuation payoff $V_B(x)$ to itself determined by the equilibrium conditions is well defined and continuous. Toward this aim, we proceed first by providing several preliminary results—Lemmas 1, 2 and 3—which we subsequently use to establish the existence in Theorem 1.

Throughout the paper, we assume that the output function is either supermodular or submodular.

Assumption 3: (SUP) *The output function $z(x, y)$ is supermodular.*

(SUB) *The output function $z(x, y)$ is submodular.*

For any price function that is continuous in seller types, the following lemma establishes that the buyer's value function $V_B(x)$ is Lipschitz continuous and differentiable.

Lemma 1: *Given A1, A3-Sup or A3-Sub, a type x buyer's value function $V_B(x)$ satisfies*

$$V_B(x) \geq \frac{1}{1-\beta} \int_M (z(x,y) - P(y) - \beta V_B(x)) f_S(y) dy$$

for any $M \subseteq [\underline{y}, \bar{y}]$. In addition, $V_B(x)$ is non-negative, increasing in x and Lipschitz continuous in equilibrium. Moreover, if price is continuous in seller types, $V_B(x)$ is a.e. differentiable. When differentiable, its derivative is

$$V_B'(x) = \frac{\int_{\mathbb{M}(x)} z_1(x,y) f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}(x)} f_S(y) dy}$$

Unless otherwise mentioned, all proofs are provided in the appendix. To conveniently identify whether a pair of agents belong to each other's matching set, we define the indicator function $d(x,y)$: $d(x,y) = 1$ if and only if $s(x,y) \geq 0$ and $d(x,y) = 0$ otherwise.

Lemma 2: *Given A1, A3-Sup or A3-Sub, for any price function $P(y, V_B)$ that is continuous in y and V_B, any Borel measurable mapping from the buyer's value functions $V_B(x)$ to the match indicator function $d(x,y)$ is continuous.*

For the existence proof, we also need to show that the endogenous distribution is continuous in the indicator function $d(x,y)$.

Lemma 3: *The mapping $d(x,y) \rightarrow (f_B(x), f_S(y))$ is well defined and continuous.*

With the above preliminary results, we are now ready to establish the existence of the equilibrium.

Theorem 1. *A search equilibrium exists in which prices are continuous in seller types.*

4.2 Definition of Sorting

The matching sets in an environment with search frictions are normally non-degenerate. It takes time for a buyer to meet a seller. With a degenerate matching set, the probability of trade within one period is zero and the buyer would almost surely experience the exit shock before any trade takes place. Therefore, he is willing to accept a range of prices offered by sellers whose types are randomly drawn from his matching set.

As a result, we can no longer use the definition of PAM (NAM) in Becker (1973). Instead, we follow the definition in Shimer and Smith (2000): for PAM, the set of mutually agreeable matches must form a lattice. More explicitly:

Definition 1: *Take $x_1 < x_2$ and $y_1 < y_2$.*

PAM: *There is PAM if $y_1 \in \mathbb{M}_B(x_1)$ and $y_2 \in \mathbb{M}_B(x_2)$ whenever $y_1 \in \mathbb{M}_B(x_2)$ and $y_2 \in \mathbb{M}_B(x_1)$.*

NAM: *There is NAM if $y_1 \in \mathbb{M}_B(x_2)$ and $y_2 \in \mathbb{M}_B(x_1)$ whenever $y_1 \in \mathbb{M}_B(x_1)$ and $y_2 \in \mathbb{M}_B(x_2)$.*

5. One-Shot Bilateral Monopoly: $\beta = 0$

In this section, we consider the case in which all of the agents face the terminal shock after one period, which is essentially a one-shot bilateral monopoly. In other words, the search frictions are at the maximal level and agents do not value the future at all ($\beta = 0$).

In this case, the surplus function $s(x, y)$ equals $z(x, y) - P(y)$, which is strictly increasing in x. If a seller of type y sets the price $P(y) = z(x, y)$, then any buyer with a type above x is willing to accept the offer. Therefore, choosing the optimal price $P(y)$ is equivalent to selecting the marginal type $x^*(y)$ to maximize the expected profit. That is,

$$\Pi(y) = \max_{\tilde{x}} \{ z(\tilde{x}, y)(1 - F_B(\tilde{x})) \}$$

Here $F_B(x) = \Gamma_B(x)$, since all buyers exit the market after one period. To make sure that $x^*(y)$ is unique, we impose the following assumption of increasing hazard rate on $\Gamma_B(x)$, which is standard in the literature.[4]

Assumption 4: $\frac{\gamma_B(x)}{1 - \Gamma_B(x)}$ *is strictly increasing in x for any $x \in [\underline{x}, \bar{x}]$.*

Theorem 2. *When $\beta = 0$, the marginal buyer type $x^*(y)$ is uniquely determined by*

$$z_1(x^*(y), y)(1 - F_B(x^*(y))) = z(x^*(y), y)f_B(x^*(y)), \text{ if } x^*(y) \in (\underline{x}, \bar{x})$$
$$x^*(y) = \underline{x}, \text{ if } z_1(\underline{x}, y)(1 - F_B(\underline{x})) \leq z(\underline{x}, y)f_B(\underline{x}) \tag{1}$$

The optimal price $P(y) = z(x^*(y), y)$. In addition, both $P(y)$ and $x^*(y)$ are continuous in y.

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[4]The assumption can be weakened if $z(x, y)$ is strictly log-concave. In that case, $\frac{\gamma_B(x)}{1 - \Gamma_B(x)}$ only need to be weakly increasing in x.

Proof: Condition (1) is obtained by taking the first order condition of the seller's objective function.

To show the uniqueness, for any given y, consider the following two functions in x,

$$L(x; y) = \frac{z_1(x, y)}{z(x, y)}$$

$$R(x) = \frac{f_B(x)}{1 - F_B(x)}$$

The function $L(x; y)$ weakly decreases in x, as the assumption of log-concavity implies $\frac{\partial L(x;y)}{\partial x} = \frac{z_{11}z - (z_1)^2}{z^2} \leq 0$. The function $R(x)$ strictly increases in x, following the assumption of strictly increasing hazard rate. Both $L(x; y)$ and $R(x)$ are continuous in x. Therefore, they can intersect at most once.

First note that $R(\bar{x})$ is always larger than $L(\bar{x}; y)$. Therefore, $\bar{x}$ can never be a marginal buyer type for any y. If $R(\underline{x}) < L(\underline{x}; y)$, then the equation $L(x; y) = R(x)$ has a unique interior solution. We can rearrange the equation and obtain the first case in condition (1). If $R(\underline{x}) \geq L(\underline{x}; y)$, then $R(x) > L(x; y)$ for any $x \in (\underline{x}, \bar{x}]$. This corresponds to the second case in condition (1).

The optimal price has been shown to equal to $z(x^*(y), y)$ in the preceding discussion. The continuity of $x^*(y)$ and $P(y)$ follows from the fact that $L(x; y)$ is continuous in y. ∎

As usual, an (interior) x^* is chosen so that the marginal revenue of increasing the marginal buyer type equals the marginal cost. The left-hand side of equation (1) represents the marginal revenue. The resulting price increment is $z_1(x^*(y), y)$ and the seller can collect this increment when trade happens with the probability of $1 - F_B(x^*(y))$. The right-hand side is the marginal cost of increasing the marginal buyer type. The seller can no longer sell to the buyers of type $x^*(y)$. The resulting loss equals the price, $z(x^*(y), y)$, times the probability of meeting a buyer of type $x^*(y)$, $f_B(x^*(y))$.

Let us now characterize sorting. Under the threshold rule and the assumptions that ensure differentiability, the definition of PAM (NAM) reduces to the condition that the derivative of the marginal type is positive (negative). That is, sorting is positive if and only if $\frac{\partial x^*(y)}{\partial y} \geq 0$ and is negative if and only if $\frac{\partial x^*(y)}{\partial y} \leq 0$. This means that we need to do a comparative static exercise to identify conditions under which this derivative is positive (or negative).

To exclude the trivial cases where $x^*(y) = \underline{x}$ for any y, we assume that $\max_y \{\frac{z_1(\underline{x}, y)}{z(\underline{x}, y)}\} > f_B(\underline{x})$.

Theorem 3. *When $\beta = 0$, sorting is always positive (PAM) if and only if the output function $z(x, y)$ is log-supermodular and sorting is always negative (NAM) if and only if $z(x, y)$ is log-submodular.*

Proof: If $x^*(y)$ is on the boundary for some y, then $\frac{\partial x^*(y)}{\partial y} = 0$. The matching is assortative.

If $x^*(y)$ is an interior solution, then from (1), it is easy to see that

$$\frac{\partial x^*(y)}{\partial y} = \frac{z_{12}[1 - F_B(x^*)] - z_2 f_B(x^*)}{2z_1 f_B(x^*) + z f'_B(x^*) - z_{11}[1 - F_B(x^*)]}$$

$$= \frac{z_{12} z - z_1 z_2}{(z_1)^2} \times \frac{1}{2 + \frac{z}{z_1} \frac{f'_B}{f_B} - \frac{z_{11} z}{(z_1)^2}} \tag{2}$$

The second equality follows after plugging in the first order condition and rearranging terms.

We also know that $\frac{1 - F_B}{f_B}$ strictly decreases in x

$$\frac{\partial}{\partial x} \left(\frac{1 - F_B}{f_B} \right) = \frac{-f_B^2 - (1 - F_B) f'_B}{f_B^2} < 0$$

$$\Rightarrow 1 + \frac{1 - F_B}{f_B} \frac{f'_B}{f_B} > 0 \Rightarrow \frac{z}{z_1} \frac{f'_B}{f_B} > -1$$

Therefore,

$$2 + \frac{z}{z_1} \frac{f'}{f} - \frac{z_{11} z}{(z_1)^2} > 2 - 1 - 1 \geq 0$$

The sorting is positive given any type distributions if and only if $z_{12} z - z_1 z_2 \geq 0$, i.e., $z(x, y)$ is log-supermodular; it is negative given any type distribution if and only if $z_{12} z - z_1 z_2 \leq 0$ i.e., $z(x, y)$ is log-submodular. ∎

The above Theorem establishes that with the maximum search frictions and information frictions, we need log-supermodularity, which is much stronger than supermodularity, to ensure positive sorting.[5] The intuition behind this result is one of the main messages of the paper and it also helps us understand the sufficiency of supermodularity when search frictions vanish. To avoid repetition, we will only focus on the PAM case in the following discussion, as the intuition for NAM is symmetric.

Consider two sellers with types y_1 and y_2, $y_2 < y_1$. Imagine that the two sellers are currently choosing the same marginal buyer type x, which is seller 2's optimal marginal type,

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[5]Notice that log-supermodularity is a stronger condition than supermodularity, as it requires z_{12} to be not only positive, but also larger than $\frac{z_1 z_2}{z}$, which itself must be strictly positive.
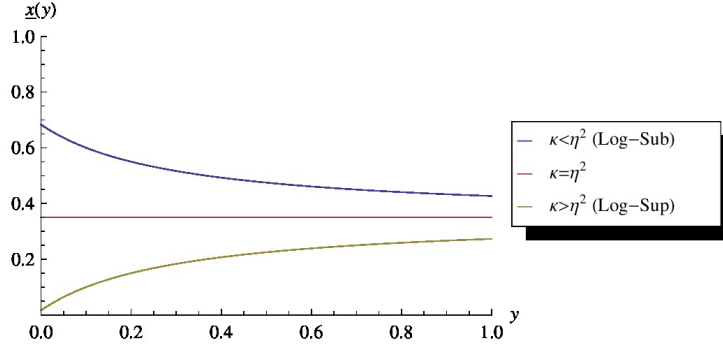
Figure 1: Seller reservation types
$$\Gamma_B(x) \sim U(0,1); \ z(x,y) = \eta\,(x+y) + xy + \kappa, \ \eta \in (0,1), \ \kappa \in (0,\eta)$$

and consider the trade-offs for each—in terms of price and trade probability—associated with increasing the marginal type slightly.

The marginal benefit of increasing the marginal type equals the price increment times the trading probability, i.e., $z_1(x,y)[1 - F_B(x)]$. Because of the supermodularity, the price increment is greater for the high type seller 1. The two sellers are currently choosing the same marginal buyer type. So their trading probabilities are the same. Overall, a high type seller has a higher marginal benefit due to supermodularity and hence faces stronger incentives to increase her marginal type buyer. We also know that the marginal cost equals the reduction in trading probability times the price, i.e., $z(x,y)f_B(x)$. The two sellers experience the same level of reduction in trading probabilities. But the high type seller loses more from the same reduction in trading probability, because the price he charges is higher. Hence, a high type seller also has a higher marginal cost and so has a stronger incentive to secure trade through lowering her marginal buyer type.

Recall that PAM requires a higher type seller to choose a higher marginal buyer type. Therefore, the increment in marginal benefit ($z_{12}(x,y)[1 - F_B(x)]$), must outweigh the increment in marginal cost ($z_2(x,y)f_B(x)$). Supermodularity is clearly insufficient. Plugging in the equilibrium condition $z_1(x,y)[1 - F_B(x)] = z(x,y)f_B(x)$, we can see that positive sorting in this case requires an output function with stronger complementarity—specifically, log-supermodularity.

In Figure 1, we plot the marginal buyer type function $x^*(y)$ with parameter specifications as shown beneath the figure. From this example, we can easily verify two conclusions we had. First, the log-supermodular condition is stronger than the supermodular: the output function $z(x,y)$ is always supermodular, but it is log-supermodular if and only if $\kappa > \eta^2$. Secondly, the sorting is positive if and only if $z(x,y)$ is log-supermodular.

To further illustrate the role of private information, let us compare the current sorting result with the one in Shimer and Smith (2000). In their setting, agents' types become observable after they are paired. In the static case, an agent is willing to trade with any type, regardless of the allocation of the bargaining power. Based on the definition of PAM (NAM), the matching is always PAM and NAM with any output function. Sellers in their paper do not face the trade-off mentioned earlier. When they increase the marginal buyer type, there is no marginal benefit (as the prices charged to other buyers remain the same), but only marginal cost, which is caused by the reduction in trading probability.

Besides the matching sets, we can also characterize how prices move with seller types. We find that prices increase in seller types if the output function exhibits a sufficiently strong combination of supermodularity and log-concavity in buyer type.

Lemma 4: $P(y)$ *increases in* y *if* $z(x,y)$ *is concave in* x *and supermodular, or if* $z(x,y)$ *is log-concave and log-supermodular.*

Proof: We know from the preceding analysis

$$\frac{\partial P(y)}{\partial y} = z_1 \frac{\partial x^*}{\partial y} + z_2$$

$$\propto z_{12}z - z_1 z_2 \frac{z_{11}z}{(z_1)^2} + z_1 z_2 \left(1 + \frac{z}{z_1}\frac{f_B'}{f_B}\right)$$

$$> z_{12}z - z_1 z_2 \frac{z_{11}z}{(z_1)^2}$$

Therefore, $\frac{\partial P(y)}{\partial y} > 0$ if $z_{12} \geq 0$ and $z_{11} \leq 0$, or if $\frac{z_{11}z}{(z_1)^2} \leq 1$ and $z_{12}z - z_1 z_2 \geq 0$. ∎

6. Frictionless Limit: $\beta \to 1$

In this section, we consider the equilibrium outcomes when the time between two consecutive periods shrinks to zero—that is, as the actual discount factor β approaches 1. We will conceptualize vanishing search frictions in this way throughout this section.

6.1 Sorting Pattern

As search frictions diminish, we find that Becker's sorting result can be restored even if information frictions remain. That is, supermodularity (submodularity) is sufficient to ensure one-to-one positive (negative) sorting in the limit. To distinguish the definition of

PAM (NAM) in Becker (1973) from the one using lattice in the current paper, we call the sorting defined in the former as perfectly positive (negative) sorting.

Theorem 4. *Given A3-SUP or A3-SUB, for any $\xi > 0$, there exists an $\epsilon > 0$ such that for any $\beta > 1 - \epsilon$,*

1. *$d(x, y) = 1$ if and only if $s(x, y) \in [0, \xi)$;*

2. *$\mu(\mathbb{M}_B(x)) \in [0, \xi)$ and $\mu(\mathbb{M}_S(y)) \in [0, \xi)$;*

3. *the matching sets converge to perfectly positive sorting if $z(x, y)$ is supermodular, i.e., there exists a strictly increasing function $m(x)$ defined on $[\underline{x}, \bar{x}]$ such that (i) $m(\underline{x}) = \underline{y}$, $m(\bar{x}) = \bar{y}$, and (ii) for any (x, y) with $d(x, y) = 1$, $\mid x - m^{-1}(y) \mid < \xi$ and $\mid y - m(x) \mid < \xi$.*

4. *the matching sets converge to perfectly negative sorting if $z(x, y)$ is submodular, i.e., there exists a strictly decreasing function $m(x)$ defined on $[\underline{x}, \bar{x}]$ such that (i) $m(\underline{x}) = \bar{y}$, $m(\bar{x}) = \underline{y}$, and (ii) for any (x, y) with $d(x, y) = 1$, $\mid x - m^{-1}(y) \mid < \xi$ and $\mid y - m(x) \mid < \xi$.*

To understand this result, note that although the sellers still face the trade-off between the price and trading probability in each period, they care less and less about the latter as they meet buyers more and more frequently. Even with a small matching set, a seller can sell almost surely before experiencing the terminal shock. Thus, a seller has incentive to raise her price so that she is only trading with buyers whose types belong to a small neighbourhood of her most preferred and feasible type.

This targeted type must be increasing in seller types. A seller would like to trade with higher type buyers, ceteris paribus, as the match value is higher. Unfortunately, to attract higher type buyers with private information, a seller would have to provide a higher utility. In the case with supermodular output functions, a higher type seller has a lower cost for providing utilities to higher type buyers, as the complementarity reduces the pressure to lower the price. Another way to understand why the targeted type must be increasing is the following. Suppose the limiting sorting pattern is not positive everywhere, i.e., there exists a buyer of type x matching with a seller of type y' and a buyer of type x' matching with a seller of type y, $x' > x$ and $y' > y$. Given that the output function is supermodular and strictly increasing in both arguments, the match surplus between x' and y' must be strictly positive. As a result, the type y' seller can match with an interval of buyers with types around x'. Then the seller can profit from asking for a higher price. The same argument can be applied to show that when the output function is submodular, the limiting sorting pattern must be negatively assortative.

The perfectly positive assortative matching also requires all buyers, including those with low types, have the chance to trade. Note that this is in general not true in the static benchmark, but it will indeed hold in the limit. Suppose some lower type buyers are not in any seller's matching set in the limit. Then these buyers exit the market only when they experience death shock, while higher type buyers exit also when they trade. These lower type buyers with slower exit rates will then congest the market. A seller will then find it profitable to lower his price slightly, as his trading probability would increase significantly.

This convergence to perfectly assortative matching may seem to be the only possible result. But it does depend on the structure of the model, as shown by the following two examples.

Example 1: Perfect Information. In the static case, we have established that it is the information friction that impedes sorting. As search frictions diminish, this effect also vanishes, as the next meeting is immediate. In this section, we show that in fact some private information is necessary for the convergence if buyers have no bargaining power.

Consider the same environment as specified before except for one departure: a buyer's type is observable to the paired seller. Based on the logic of the Diamond paradox, all buyers have zero continuation payoff. The matching set then must be characterized by a cut-off rule. That is, a seller trades with any buyer whose type is above a certain threshold. Therefore, the matching is never perfectly sorted and the search equilibrium fails to converge to perfect competition.

This example suggests that convergence requires some underlying factors that ensure positive payoff of buyers. It could be the private information as in the current model, or it might be some right allocation of bargaining power. We will leave the general characterization of the sufficient conditions for convergence (in terms of information set, bargaining protocols, etc.) to future works.

Example 2: Exits are Replaced with Clones. Instead of having a fixed measure of entrants with exogenously given type distributions, suppose agents who exit are replaced by their clones so that the exogenously given type distributions of incumbents are preserved over time. Then some low type buyers will never trade and the sorting pattern in the limit does not become perfectly assortative in general. The reason is that even if some lower type buyers exist who never trade, their population is still only an exogenously given fraction of the market size and will not grow due to low exit rates. Then sellers do not have incentive to trade with these low type buyers in equilibrium.

6.2 Convergence to Perfect Competition

We have established in the previous section that—as in the Walrasian benchmark—the condition for (perfect) sorting approaches supermodularity as frictions vanish. we show in this section that the equilibrium price function and the sorting pattern governed by $m(x)$ also converge to their competitive limits.

Theorem 5. *Given A3-Sup or A3-Sub, equilibrium prices converge pointwise to the price function $P^*(y)$, where*

$$P^*(y) = z(\underline{x}, \underline{y}) + \int_{\underline{y}}^{y} z_2(m^{-1}(\tilde{y}), \tilde{y})d\tilde{y}, \text{ if } z(x,y) \text{ is supermodular,}$$

$$P^*(y) = z(\underline{x}, \bar{y}) - \int_{y}^{\bar{y}} z_2(m^{-1}(\tilde{y}), \tilde{y})d\tilde{y}, \text{ if } z(x,y) \text{ is submodular,}$$

The above theorem shows that the equilibrium price approaches to its Walrasian counterpart, and each player gets her marginal contribution in the limit. Because the buyer in a match can meet another seller almost immediately, the seller in the match faces almost perfect competition from other sellers. Thus, in the limit, the marginal price increment associated with each seller type approaches that type's marginal contribution to the output.

With the preceding results regarding the perfect sorting and equilibrium price, we still cannot conclude that the equilibrium converges to the perfect competitive equilibrium. What still needs to be shown is that the $m(x)$ function, which governs the sorting pattern, converges to the competitive limit. In addition, the convergence requires no delay in trade. The time to trade depends on both the probability of trade per-period and the time between two consecutive periods. We know from Theorem 4 that the trading probability per-period converges to zero as the time between two periods shrinks to zero. Therefore, the expected duration before trade depends on which one of them converges to zero at a faster rate. The following lemma shows that the search frictions diminish at a faster speed.

Lemma 5: *As β converges to one, the expected time that it takes to trade for all the buyers and sellers approaches zero.*

To understand why the time to trade converges to zero, let us consider the decision problem of a seller. When the search friction is very small, we know that the seller has already chosen a price such that the matching surplus is close to zero. Suppose the time to trade is still positive and there is a further reduction in the search friction. The seller can benefit from a lower search friction in two ways. He can either keep the same price and

reduce the time to trade, or raise the price and keep the same time to trade. Notice that the price increment is close to zero while the reduction in time to trade is strictly positive. Therefore, the seller will respond by reducing the time to trade. In other words, search frictions must diminish at a faster speed than the per-period trading probabilities do. The result also applies to buyers due the nature of bilateral trade.

In addition, the convergence requires that the limiting matching set becomes efficient. In a perfectly competitive market with no information friction, buyer entrants and seller entrants with the same percentile types match with each other if the output function is supermodular. If it is submodular, then they match if their percentiles add up to one. The following lemma shows that the limiting matching set has the same property.

Lemma 6: *As β converges to one, $\Gamma_B(x) - \Gamma_S(m(x))$ converges to zero if $z(x,y)$ is supermodular and $\Gamma_B(x) + \Gamma_S(m(x))$ converges to one if $z(x,y)$ is submodular.*

This result is obtained based on two preliminary conclusions. First, even after taking into account the change in the market size, the measure of forced exit per-period still shrinks to zero. Second, the sorting converges to perfect sorting. Then the total measure of buyer exits with types below x is arbitrarily close to the total measure of seller exits with types below $m(x)$ under supermodularity. We can then use the steady state condition to obtain the result in the lemma.

We are now ready to establish the convergence result.

Theorem 6. *As β converges to one, a search equilibrium converges to perfect competition.*

To further understand this convergence result, we utilize the insights from Gretsky, Ostroy, and Zame (1999), i.e. interpreting "perfect competition" as *the inability of individuals to (favorably) influence prices.* More precisely, this will entail sellers facing perfectly elastic demand and buyers facing perfectly elastic supply. Toward formalizing these concepts, let us introduce the notion of a "price elasticity of demand" faced by a type y seller setting price p:

$$E_y\left(p\right) = \frac{\left(\frac{\partial(\mu(\mathbb{M}_S(y,p)))}{\mu(\mathbb{M}_S(y,p))}\right)}{\left(\frac{\partial p}{p}\right)}$$

Naturally, this reflects the responsiveness of a seller's trading probability (per period) to her chosen price.

Accordingly, the term "value elasticity of supply" will pertain to the corresponding notion

for a type x buyer with outside option value v:

$$E_x\left(v\right) = \frac{\left(\frac{\partial(\mu(\mathbb{M}_B(x,v)))}{\mu(\mathbb{M}_B(x,v))}\right)}{\left(\frac{\partial v}{v}\right)}$$

In this case, let us imagine that each buyer is choosing an optimal matching set, which is determined implicitly by her choice of an option value (taking prices as given). Following this logic, the above object reflects how a buyer's per-period trading probability responds to this option value. [6]

For agents to be unable to influence prices, the two objects above should be arbitrarily large. The following lemma shows that this is indeed the case.

Lemma 7: *Given A3-SUP or A3-SUB, the price elasticity of demand faced by any seller $E_y\left(p\right)$ and the value elasticity of supply faced by any buyer $E_x\left(v\right)$ are perfectly elastic.*

7. Conclusion

The presence of buyer private information does impede sorting, and we have highlighted the relationship between the strength of this effect and the degree of competition in the market. At one extreme, when there is bilateral monopoly power in each buyer-seller meeting, PAM requires a log-supermodular production function, which is of course a stronger condition than standard supermodularity. Higher types also have higher opportunity costs for failing to trade, so the added incentives to ensure trade takes place are in conflict with sorting.

These incentives remain relevant in a dynamic frictional setting, but they grow inconsequential as we approach the frictionless limit. Thus, as search frictions vanish, the sorting consequences of private information do as well, and the standard supermodularity condition is sufficient to generate positive sorting.

Moreover, given the sorting pattern, the increasing competition results a division of the surplus that approaches its competitive counterpart: agents from both sides of the market obtain their marginal contributions. The limiting sorting pattern itself also coincides with that of perfect competition and no agents expect delay in trade in the limit. We show that

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[6]If the product is homogeneous, the standard definition of elasticity of supply can be redefined as the percentage change in quantity supplied divided by the percentage change in consumer surplus, i.e., the difference between willingness to pay and price. In this model, it is easier to use this alternative definition, as sellers are heterogeneous, which leads to an infinite number of willingness to pay and price for each buyer, while a buyer only needs to choose one option value, which corresponds to the consumer surplus in the world with homogeneous products.

with information frictions and the bargaining protocol that sellers make take-it-or-leave-it offers, the search equilibria converge to perfect competition.

Appendix

Proof of Lemma 1

Proof:

Step 1: Inequality for $V_B(x)$ and non-negativity of $V_B(x)$.

First, $V_B(x)$ can be rearranged and expressed as

$$V_B(x) = \frac{1}{1-\beta} \int_{\mathbb{M}_B(x)} (z(x,y) - P(y) - \beta V_B(x)) f_S(y) dy$$

Any $M \neq \mathbb{M}_B(x)$ either excludes $y \in \mathbb{M}_B(x)$, in which case $z(x,y) - P(y) - \beta V_B(x) > 0$, or includes $y \notin \mathbb{M}_B(x)$, in which case $z(x,y) - P(y) - \beta V_B(x) < 0$. The inequality therefore follows.

$V_B(x)$ is non-negative given that any $y \in \mathbb{M}_B(x)$ satisfies $s(x,y) \geq 0$.

Step 2: $V_B(x)$ is increasing in x.

Consider any $x_2 \geq x_1$,

$$(1-\beta)[V_B(x_2) - V_B(x_1)]$$
$$= \int_{\mathbb{M}_B(x_2)} (z(x_2,y) - P(y) - \beta V_B(x_2)) f_S(y) dy - \int_{\mathbb{M}_B(x_1)} (z(x_1,y) - P(y) - \beta V_B(x_1)) f_S(y) dy$$
$$\geq \int_{\mathbb{M}_B(x_1)} [z(x_2,y) - z(x_1,y) - \beta(V_B(x_2) - V_B(x_1))] f_S(y) dy$$

$$\Rightarrow V_B(x_2) - V_B(x_1) \geq \frac{\int_{\mathbb{M}_B(x_1)} [z(x_2,y) - z(x_1,y)] f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}_B(x_1)} f_S(y) dy} \geq 0$$

Step 3: $V_B(x)$ is Lipschitz continuous.

Similar to step 2, we obtain the following inequality

$$V_B(x_2) - V_B(x_1) \leq \frac{\int_{\mathbb{M}_B(x_2)} [z(x_2,y) - z(x_1,y)] f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}_B(x_2)} f_S(y) dy}$$

By the assumption that $z(x, y)$ is Lipschitz continuous, there exists a real constant κ such that $\mid z(x_2, y) - z(x_1, y) \mid \leq \kappa(x_2 - x_1)$. Combined with the above two inequalities

$$\frac{-\kappa(x_2 - x_1) \int_{\mathbb{M}_B(x_1)} f_S(y)dy}{1 - \beta + \beta \int_{\mathbb{M}_B(x_1)} f_S(y)dy} \leq V_B(x_2) - V_B(x_1) \leq \frac{\kappa(x_2 - x_1) \int_{\mathbb{M}_B(x_2)} f_S(y)dy}{1 - \beta + \beta \int_{\mathbb{M}_B(x_2)} f_S(y)dy}.$$

Therefore, $\mid V_B(x_2) - V_B(x_1) \mid \leq \kappa(x_2 - x_1)$, which implies that $V_B(x)$ is Lipschitz continuous.

Step 4: $V_B(x)$ is a.e. differentiable.

Step 4.1: $\mathbb{M}_B(x)$ is u.h.c. and a.e. l.h.c..

Step 4.1.1: $\mathbb{M}_B(x)$ is u.h.c..

Take any sequence $(x_n, y_n) \to (x, y)$ with $y_n \in \mathbb{M}_B(x_n)$ for any n, i.e., $z(x_n, y_n) - \beta V_B(x_n) - P(y_n) \geq 0$ for all n. In the limit, $z(x, y) - \beta V_B(x) - P(y) \geq 0$ because $z(x, y)$, $V_B(x)$ and $P(y)$ are continuous. This implies $y \in \mathbb{M}(x)$.

Step 4.1.2: the surplus function $s(x, y)$ is rarely constant in one variable.

Define $N_s(x) = \{y : s(x, y) = 0\}$, $N_s(y) = \{x : s(x, y) = 0\}$ and $N_s = \{(x, y) : s(x, y) = 0\}$. Pick $x \neq x'$ and $y \neq y'$, such that $s(x, y) = s(x', y) = s(x, y') = 0$. If A2-Sup or A2-Sub is satisfied, it must be true that $s(x', y') \neq 0$. To see this, notice

$$s(x, y) - s(x', y) = z(x, y) - z(x', y) - \beta(V_B(x) - V_B(x'))$$
$$s(x, y') - s(x', y') = z(x, y') - z(x', y') - \beta(V_B(x) - V_B(x'))$$

Because $z(x, y) - z(x', y) \neq z(x, y') - z(x', y')$, $0 = s(x, y) - s(x', y) \neq s(x, y') - s(x', y')$, which implies $s(x', y') \neq 0$.

Then by the proof in Appendix B of Shimer and Smith (2000), $\mu(N_s(x)) = 0$ for a.e. x, $\mu(N_s(y)) = 0$ for a.e. y and $\mu(N_s) = 0$ a.e..

Step 4.1.3: $\mathbb{M}_B(x)$ is a.e. l.h.c..

Take any sequence $x_n \to x$ and any $y \in \mathbb{M}_B(x)$. First consider the scenario in which there exists a subsequence x_m of x_n, such that for any x_m,

$$\sup_{\hat{y}}\{z(x_m, \hat{y}) - \beta V_B(x_m) - P(\hat{y})\} \geq z(x, y) - \beta V_B(x) - P(y) \geq \inf_{\hat{y}}\{z(x_m, \hat{y}) - \beta V_B(x_m) - P(\hat{y})\}$$

By the continuity of $z(x, y)$ and $P(y)$, there exists at least one y_m that satisfies,

$$z(x_m, y_m) - \beta V_B(x_m) - P(y_m) = z(x, y) - \beta V_B(x) - P(y)$$

If there are multiple solutions, pick the one that is the closest to y. This defines a sequence $\{y_m\}$. Clearly, $y_m \in \mathbb{M}_B(x_m)$. We need to show that for almost all x, there exists a subsequence y_k of y_m, such that $y_k \to y$.

We know a convergent subsequence always exists because $\{y_m\}$ is bounded.

Case 1. We first consider the case where there exists $\epsilon > 0$ and $K > 0$ such that for any $k > K$ and any $\hat{y} \in [y - \epsilon, y + \epsilon]$, either $\hat{y} \notin \mathbb{M}_B(x_k)$ or $s(x, \hat{y}) = 0$. In this scenario, it is possible that none of the subsequences converge to y. However, the measure of $\{x, y\}$ that satisfies this condition is 0.

Case 2. Next, we consider all of the complementary scenarios, that is, for any $\epsilon > 0$ and $K > 0$, there exist $k > K$ and $\hat{y} \in [y - \epsilon, y + \epsilon]$, such that $\hat{y} \in \mathbb{M}_B(x_k)$.

Suppose none of the subsequences converge to y. Since we have excluded the two situations in case 1, the non-convergence implies that there exists $\eta > 0$ and $K_1 > 0$, such that for any $k > K_1$

$$| z(x_k, y) - z(x, y) - \beta(V_B(x_k) - V_B(x)) | > 2\eta$$

On the other hand, by the continuity of z and V_B, there exist K_2, such that for all $k > K_2$

$$| z(x_k, y) - z(x, y) - \beta(V_B(x_k) - V_B(x)) |$$
$$\leq | z(x_k, y) - z(x, y) | + \beta | V_B(x_k) - V_B(x) | < 2\eta$$

Pick $K = \max\{K_1, K_2\}$. The above two inequalities cannot hold at the same time. This is a contradiction.

Finally, consider scenarios where such subsequence x_m does not exist. That is, for any subsequence x_m, either $\sup_{\hat{y}}\{z(x_m, \hat{y}) - \beta V_B(x_m) - P(\hat{y})\} < z(x, y) - \beta V_B(x) - P(y)$ or $\inf_{\hat{y}}\{z(x_m, \hat{y}) - \beta V_B(x_m) - P(\hat{y})\} > z(x, y) - \beta V_B(x) - P(y)$. Here we show the proof for the first case. The proof for the second case is similar and is skipped.

Define $y_m \in \text{argmax}\{z(x_m, \hat{y}) - \beta V_B(x_m) - P(\hat{y})\}$. If there is more than one argmax, pick the one that is the closest to y. Based on this construction, we can then follow the proof in case 2 to show that any convergent subsequence of y_m must converge to y.

Step 4.2: Decomposition for the slope of $V_B(x)$

Consider any x such that $\mathbb{M}_B(x)$ is continuous. Take any sequence $x_n \to x$, for each n,

$$(1-\beta)\frac{V_B(x_n) - V_B(x)}{x_n - x} = \int_{\mathbb{M}_B(x_n)-\mathbb{M}_B(x)} \frac{z(x_n, y) - P(y) - \beta V_B(x_n)}{x_n - x} f_S(y) dy$$
$$+ \int_{\mathbb{M}_B(x)} [\frac{z(x_n, y) - z(x, y)}{x_n - x} - \beta\frac{V_B(x_n) - V_B(x)}{x_n - x}] f_S(y) dy$$

Take the limit as $n \to \infty$. The first integral vanishes because 1) $\mathbb{M}_B(x)$ is continuous a.e., and 2) the surplus function vanishes at changes in $\mathbb{M}_B(x)$. Rearranging terms, we get the proposed derivative.

∎

Proof of Lemma 2

Proof: We have proved that the surplus function is rarely constant in one variable in step 4.1.2 of the proof for lemma 1. Define the set $\sum_s(\eta) = \{(x,y) : |s(x,y)| \in [0,\eta]\}$. This set shrinks monotonically to $\cap_{k=1}^{\infty} \sum_s(1/k) = N_s$.

$$\lim_{\eta \to 0} (\mu \times \mu)(\sum_s(\eta)) = (\mu \times \mu)(\cap_{k=1}^{\infty} \sum_s(1/k)) = (\mu \times \mu)(N_s) = 0$$

Let V_B^1 and V_B^2 be two value functions, and d^1 and d^2 be the corresponding match indicator functions.

Since $P(y, V_B)$ is continuous in V_B, for any $\epsilon > 0$, there exists a $\eta' > 0$, such that

$$\beta \parallel V_B^1(x) - V_B^2(x) \parallel < \eta' \Rightarrow | P(y, V_B^1) - P(y, V_B^2) | < \epsilon, \text{ for any } y$$

In other words, we can always pick close enough value functions such that the price functions are close. Let $\eta = 2\max\{\eta', \epsilon\}$.

If $s^1(x,y) = z(x,y) - \beta V_B^1(x) - P(y, V_B^1) > \eta$, then $s^2(x,y) = z(x,y) - \beta V_B^2(x) - P(y, V_B^2) > 0$. So $d^1(x,y) = d^2(x,y) = 1$. By the same logic, If $s^1(x,y) < -\eta$, then $s^2(x,y) < 0$. So $d^1(x,y) = d^2(x,y) = 0$. As a result, $\{(x,y) : d^1(x,y) \neq d^2(x,y)\} \subseteq \sum_{s^1}(\eta)$. The Lebesgue measure of $\sum_{s^1}(\eta)$ vanishes as $\eta \to 0$. The continuity is thus established

$$\lim_{\|V_B^1(x) - V_B^2(x)\| \to 0} \parallel d^1(x,y) - d^2(x,y) \parallel_{\mathcal{L}^1} = 0.$$

∎

Proof of Lemma 3

Proof:

Step 1: The mapping is well defined.

Given entrant densities $\gamma_B(x)$ and $\gamma_S(y)$, the mapping is well defined if there exist unique functions $\hat{f}_B$ and $\hat{f}_S$ that solve the following system of equations,

$$\hat{f}_B(x) = \frac{\gamma_B(x)}{1 - \beta + \beta \frac{\int d(x,y)\hat{f}_S(y)dy}{\int \hat{f}_S(y)dy}}$$

$$\hat{f}_S(y) = \frac{\gamma_S(y)}{1 - \beta + \beta \frac{\int d(x,y)\hat{f}_B(x)dx}{\int \hat{f}_B(x)dx}}$$

It is easy to see that $\hat{f}_B(x) \in [\gamma_B(x), \gamma_B(x)/(1-\beta)]$ and $\hat{f}_S(y) \in [\gamma_S(y), \gamma_S(y)/(1-\beta)]$.

One can apply a log transformation method similar to that used in Shimer and Smith (2000) and reformulate the problem as a fixed-point problem.

$$\Phi_B(h) = \log \frac{\gamma_B(x)}{1 - \beta + \beta \frac{\int d(x,y)e^{h_S(y)}dy}{\int e^{h_S(y)}dy}}$$

$$\Phi_S(h) = \log \frac{\gamma_S(y)}{1 - \beta + \beta \frac{\int d(x,y)e^{h_B(x)}dx}{\int e^{h_B(x)}dx}}$$

where $h_B(x) = \log \hat{f}_B(x)$, $h_S(y) = \log \hat{f}_S(y)$, $h = (h_B, h_S)'$. The mapping is well defined if $\Phi(h) = h$ has a unique fixed point. We prove this using the Contraction Mapping Theorem. Consider h^1 and h^2,

$$\Phi_B(h^2) - \Phi_B(h^1) = \log \frac{1 - \beta + \beta \frac{\int d(x,y)e^{h_S^1(y)}dy}{\int e^{h_S^1(y)}dy}}{1 - \beta + \beta \frac{\int d(x,y)e^{h_S^2(y)}dy}{\int e^{h_S^2(y)}dy}}$$

$$\leq \log \frac{1 - \beta + \beta e^{\|h_S^1 - h_S^2\|} \frac{\int d(x,y)e^{h_S^2(y)}dy}{\int e^{h_S^2(y)}dy}}{1 - \beta + \beta \frac{\int d(x,y)e^{h_S^2(y)}dy}{\int e^{h_S^2(y)}dy}}$$

$$\leq \log \frac{1 - \beta + \beta e^{\|h_S^1 - h_S^2\|}}{1 - \beta + \beta}$$

$$= \log[1 - \beta + \beta e^{\|h_S^1 - h_S^2\|}]$$

The first inequality follows because $e^{\|h_S^1 - h_S^2\|} > e^{h_S^1(y) - h_S^2(y)}$ for any y while the second inequality holds because $e^{\|h_S^1 - h_S^2\|} > 1$ and hence the term is increasing in $\frac{\int d(x,y)e^{h_S^2(y)}dy}{\int e^{h_S^2(y)}dy}$. We

thus have

$$\frac{\Phi_B(h^2) - \Phi_B(h^1)}{\parallel h^1_S - h^2_S \parallel} \leq \frac{\log[1 - \beta + \beta e^{\parallel h^1_S - h^2_S \parallel}]}{\parallel h^1_S - h^2_S \parallel}$$

In addition, we know that $h_S(y) \in [\log(\gamma_S(y)), \log(\gamma_S(y)) - \log(1-\beta)]$, which implies $\parallel h^1_S - h^2_S \parallel \in [0, -\log(1-\beta)]$. Since the right hand side of the above inequality increases in $\parallel h^1_S - h^2_S \parallel$,

$$\frac{\Phi_B(h^2) - \Phi_B(h^1)}{\parallel h^1_S - h^2_S \parallel} \leq \frac{\log[1 - \beta + \frac{\beta}{1-\beta}]}{\log\frac{1}{1-\beta}} = \chi \in (0,1)$$

The same argument applies in the other direction and we thus obtain

$$\frac{\parallel \Phi_B(h^1) - \Phi_B(h^2) \parallel}{\parallel h^1_S - h^2_S \parallel} \leq \chi$$

We have the symmetric inequality for y. Denote $\Phi(h) = (\Phi_B(h), \Phi_S(h))'$, combining the two inequalities,

$$\parallel \Phi(h^1) - \Phi(h^2) \parallel \leq A \parallel h^1 - h^2 \parallel$$

where A is a matrix with $\mid A \mid = -\chi^2 \in (-1, 1)$. We have thus proven that it is a contraction mapping.

Step 2: The mapping is continuous.

Define $G_B(d, \hat{f})(x) = \hat{f}_B(x)[1 - \beta + \beta \int d(x,y)\frac{\hat{f}_S(y)}{\int \hat{f}_S(y)dy}dy] - \gamma_B(x)$ and $G_S(d, \hat{f})(y) = \hat{f}_S(y)[1 - \beta + \beta \int d(x,y)\frac{\hat{f}_B(x)}{\int \hat{f}_B(x)dx}dx] - \gamma_S(y)$, $G(d, \hat{f}) = (G_B(d, \hat{f}), G_S(d, \hat{f}))$. In equilibrium, $G(d, \hat{f}) = 0$

Suppose that there exist d^1 and d^2 with $\parallel d^1 - d^2 \parallel_{\mathcal{L}^1} \to 0$, such that $\parallel \hat{f}^1 - \hat{f}^2 \parallel_{\mathcal{L}^1} \nrightarrow 0$. Then there exists $\hat{\epsilon} > 0$ such that $\parallel G(d^1, \hat{f}^2) \parallel_{\mathcal{L}^1} > \hat{\epsilon}$. WOLG, assume $\parallel G_B(d^1, \hat{f}^2) \parallel_{\mathcal{L}^1} > \epsilon$.

On the other hand,

$$\parallel G_B(d^1, \hat{f}^2) \parallel_{\mathcal{L}^1} = \parallel G_B(d^1, \hat{f}^2) - G_B(d^2, \hat{f}^2) \parallel_{\mathcal{L}^1}$$

$$= \parallel \hat{f}^2_B(x)\beta \int (d^1(x,y) - d^2(x,y))\frac{\hat{f}^2_S(y)}{\int \hat{f}^2_S(y)dy}dy \parallel_{\mathcal{L}^1} < \epsilon$$

The last line follows since $\frac{\hat{f}^2_S(y)}{\int \hat{f}^2_S(y)dy}$ and $\hat{f}^2_B(x)$ are bounded for any x and y. This leads to

a contradiction.

∎

Proof of Theorem 1

Proof:

Equilibrium exists if $T(V_B) = V_B$ has a unique fixed point, where,

$$T(V_B) = \int \max\{z(x,y) - P(y, V_B), \beta V_B(x)\} f_S^{V_B}(y) dy$$

Following the Schauder Fixed Point Theorem, we need a nonempty, closed, bounded and convex domain ψ such that,

1. $T : \psi \to \psi$.

2. $T(\psi)$ is an equicontinuous family.

3. T is a continuous operator.

Let ψ be the space of Lipschitz continuous functions V_B on $[\underline{x}, \bar{x}]$, with lower bound 0 and upper bound $\sup_{x,y} z(x, y)$. Clearly, ψ is nonempty, closed, bounded and convex. Next, we check that the above three points hold.

Step 1: $T : \psi \to \psi$ and $T(\psi)$ is an equicontinuous family.

Take any x_1 and x_2 with $x_1 \neq x_2$

$$| TV_B(x_2) - TV_B(x_1) |$$
$$\leq \int | \max\{z(x_2,y) - P(y, V_B), \beta V_B(x_2)\} - \max\{z(x_1, y) - P(y, V_B), \beta V_B(x_1)\} | f_S^{V_B}(y) dy$$
$$\leq \int | \max\{z(x_2, y) - z(x_1, y), \beta(V_B(x_2) - V_B(x_1))\} | f_S^{V_B}(y) dy$$

Since both $z(x, y)$ and $V_B(x)$ are Lipschitz-continuous, $T(\psi)$ is Lipschitz-continuous, which implies equicontinuous. This also establishes that T is a mapping from ψ to ψ.

Step 2: T is continuous.

Take any V_B^1 and V_B^2 with $V_B^2 \neq V_B^1$ in ψ. For any x,

$$| TV_B^2(x) - TV_B^1(x) |$$

$$= | \int \max\{z(x,y) - P(y,V_B^2), \beta V_B^2(x)\} f_S^{V_B^2}(y)dy - \int \max\{z(x,y) - P(y,V_B^1), \beta V_B^1(x)\} f_S^{V_B^1}(y)dy |$$

$$\leq | \int \max\{z(x,y) - P(y,V_B^2), \beta V_B^2(x)\} f_S^{V_B^2}(y)dy - \int \max\{z(x,y) - P(y,V_B^2), \beta V_B^2(x)\} f_S^{V_B^1}(y)dy |$$

$$+ | \int (\max\{z(x,y) - P(y,V_B^2), \beta V_B^2(x)\} - \max\{z(x,y) - P(y,V_B^1), \beta V_B^1(x)\}) f_S^{V_B^1}(y)dy |$$

$$= D_1(x) + D_2(x)$$

For $D_1(x)$

$$D_1(x) \leq \int \max\{z(x,y) - P(y,V_B^2), \beta V_B^2(x)\} \mid f_S^{V_B^2}(y) - f_S^{V_B^1}(y) \mid dy$$

$$\leq \sup_{x,y}[\max\{z(x,y) - P(y,V_B^2), \beta V_B^2(x)\}] \int \mid f_S^{V_B^2}(y) - f_S^{V_B^1}(y) \mid dy$$

Since $f_S(y)$ is continuous in V_B, as $\| V_B^2 - V_B^1 \| \to 0$, $D_1(x) \to 0$.

For $D_2(x)$

$$D_2(x) \leq \int \mid \max\{z(x,y) - P(y,V_B^2), \beta V_B^2(x)\} - \max\{z(x,y) - P(y,V_B^1), \beta V_B^1(x)\} \mid f_S^{V_B^1}(y)dy$$

$$\leq \int \mid \max\{P(y,V_B^1) - P(y,V_B^2), \beta V_B^2(x) - \beta V_B^1(x)\} \mid f_S^{V_B^1}(y)dy$$

Since $P(y,V_B)$ is continuous in V_B, $D_2(x) \to 0$ as $\| V_B^2 - V_B^1 \| \to 0$.

Step 3: Verify that there exists at least one price function that is continuous in y and V_B in equilibrium.

Seller's problem is $\max_p \Omega(y,p,V_B)$, where

$$\Omega(y,p,V_B) = p \frac{\int_{\mathbb{M}_S(y;p,V_B)} f_B(x)dx}{1 - \beta + \beta \int_{\mathbb{M}_S(y;p,V_B)} f_B(x)dx}$$

Given that $V_B(x)$ and $z(x,y)$ are continuous, the matching set $\mathbb{M}_S(y;p,V_B)$ is continuous in y, p and V_B. As a result, $\Omega(y,p,V_B)$ is continuous in those three arguments. In addition, $p \in [0, \sup_{x,y}\{z(x,y)\}]$, which is compact valued. By the Maximum Theorem, $P(y,V_B)$ is u.h.c. in y and V_B.

Next, we show that the function $\Omega(y, p, V_B)$ is strictly concave in p and hence $P(y, V_B)$ is single valued. Combined with the u.h.c. property, $P(y, V_B)$ must be continuous.

To simplify the notations, use $Q(p; y, V_B)$ to denote the trading probability given price p, i.e., $Q(p; y, V_B) = \int_{\mathbb{M}_S(y; p, V_B)} f_B(x) dx$. It is easy to verify that $Q(p; y, V_B)$ is twice differentiable in p, with $Q'(p; y, V_B) < 0$ and $Q''(p; y, V_B) = 0$. Therefore, $\Omega(y, p, V_B)$ is also twice differentiable and

$$\frac{\partial^2 \Omega(y, p, V_B)}{\partial p^2} = \frac{2Q'(p; y, V_B)[1 - \beta]}{(1 - \beta + \beta Q)^2}(1 - \frac{p\beta Q'(p; y, V_B)}{1 - \beta + \beta Q}) < 0$$

This shows that the function $\Omega(y, p, V_B)$ is strictly concave in p. ∎

Proof of Theorem 4

Proof:

Step 1: $d(x, y) = 1 \iff s(x, y) \in [0, \xi)$ for any $\xi > 0$.

The direction "$\Leftarrow$" follows from the construction of function $d(x, y)$.

To see the other direction, notice that $d(x, y) = 1$ implies $s(x, y) \geq 0$. Suppose that there exists a $\tilde{\xi} > 0$, an $\tilde{x} \in [\underline{x}, \bar{x}]$ and a $\tilde{y} \in [\underline{y}, \bar{y}]$ such that $s(\tilde{x}, \tilde{y}) = z(\tilde{x}, \tilde{y}) - P(\tilde{y}) - \beta V_0(\tilde{x}) > \tilde{\xi}$ for any $\beta \in [0, 1)$. Since the function $s(x, y)$ is Lipschitz continuous, $s(\tilde{x}, \tilde{y}) > \tilde{\xi}$ implies that $\mathbb{M}_S(\tilde{y})$ is non-empty.

We can divide this seller's matching set into two parts, $\mathbb{M}_S^1(\tilde{y}) = \{x : s(x, \tilde{y}) \geq \tilde{\xi}_1\}$ and $\mathbb{M}_S^2(\tilde{y}) = \{x : s(x, \tilde{y}) \in [0, \tilde{\xi}_1]\}$ with some $\tilde{\xi}_1 < \tilde{\xi}$, such that the two matching sets have the same probability measure. Denote the probability measure of $\mathbb{M}_S^i(\tilde{y})$ as Q $(i = 1, 2)$.

Seller $\tilde{y}$'s expected profit can then be written as

$$\Pi(\tilde{y}) = \frac{2QP(\tilde{y})}{1 - \beta + 2\beta Q} = \frac{P(\tilde{y})}{1 + \frac{1 - 2Q}{2Q}(1 - \beta)}$$

If the seller raises the price by $\tilde{\xi}_1$, then the new matching set becomes $\mathbb{M}_S^1(\tilde{y})$. It is profitable for the seller to raise the price if the change in profit is positive, i.e.,

$$\tilde{\xi}_1[1 + \frac{1 - 2Q}{2Q}(1 - \beta)] - \frac{1}{2Q}(1 - \beta)P(\tilde{y}) > 0$$

This inequality holds in the limit if $\frac{1-\beta}{Q}$ converges to zero. Denote the market size as M and the average trading probability of buyer as $\bar{\mu}(\mathbb{M}_B)$.[7] The stationary condition

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[7]Both variables depend on β. Here we abuse the notation and simply write them as M and $\bar{\mu}(\mathbb{M}_B)$.

$1 = M[1 - \beta + \beta\bar{\mu}(\mathbb{M}_B)]$ implies that $M[1 - \beta + \beta\mu(\mathbb{M}_B(x))]$ is finite or zero for a.e. x. In addition, we know

$$\frac{f_B(x)}{1 - \beta} = \frac{\gamma_B(x)}{1 - \beta} \frac{1}{M[1 - \beta + \beta\mu(\mathbb{M}_B(x))]}$$

The right-hand side of the above equation diverges to infinity. Therefore, $\frac{f_B(x)}{1-\beta}$ also diverges to infinity. As a result, $\frac{1-\beta}{Q}$ converges to zero. We have proved the claim.

The proof of perfect negative assortative matching under submodularity is essentially the same and thus is skipped.

Step 2: $\mu(\mathbb{M}_B(x)) \in [0, \xi)$ and $\mu(\mathbb{M}_S(y)) \in [0, \xi)$.

We show that the statement holds in the following two cases: 1) the surplus function $s(x, y)$ is not always zero over the matching set and 2) $s(x, y)$ is zero over the matching set.

Step 2.1: when $s(x, y)$ is not always zero over the matching set.

Suppose there exist a $\tilde{\xi} > 0$ and a $\tilde{y} \in [\underline{y}, \bar{y}]$ such that for any $\epsilon \in (0, 1)$, $\mu(\mathbb{M}_S(\tilde{y})) > \tilde{\xi}$ for some $\beta \in [1 - \epsilon, 1)$.

Since $\mu(\mathbb{M}_S(\tilde{y})) > \tilde{\xi}$ and the surplus function $s(x, y)$ is continuous and not always zero, the seller can always raise the price and obtain a new matching set with a strictly positive trading probability. Following the same argument in the first step, the seller would find it profitable to raise the price when β is large enough. This is a ontradiction.

Following the same reasoning, $\mu(\mathbb{M}_B(x)) \in [0, \xi)$ for any $\xi > 0$.

Step 2.2: when $s(x, y)$ is zero over the matching set.

Suppose that there exist an $\tilde{x}$ and a $\xi_1 > 0$, such that for any $\epsilon \in (0, 1)$, the matching set $\mathbb{M}_B(\tilde{x})$ has a subset $[\tilde{y}_1, \tilde{y}_2]$ with $\tilde{y}_2 - \tilde{y}_1 \geq \xi_1$ for some $\beta \in [1 - \epsilon, 1)$.

Denote y_m as the middle point of the interval, $y_m = \frac{\tilde{y}_1 + \tilde{y}_2}{2}$. We first show that there exist an $\hat{\epsilon} \in (0, 1)$ such that for any $\beta \in [1 - \hat{\epsilon}, 1)$, $\tilde{x}$ is the unique element of $\mathbb{M}_S(y_m)$. The following proof is based on the assumption that $z(x, y)$ is supermodular. The same argument applies to submodular output functions and is hence skipped.

Consider any $x_l < \tilde{x}$. Suppose $x_l \in \mathbb{M}_S(y_m)$, i.e., $s(x_l, y_m) \geq 0$. Define $y' = y_m - \frac{\xi_1}{2}$. This y' is in the interval $[\tilde{y}_1, \tilde{y}_2]$ because the length of the interval is greater than ξ_1. Therefore $s(\tilde{x}, y') = 0$. By supermodularity, any $y \in [y', y_m]$ must be an element in $\mathbb{M}_B(x_l)$. We know from lemma 1 that the surplus function is rarely constant. Therefore, the result in step 2.1

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applies to $\mathbb{M}_B(x_l)$. However, we know $[y', y_m] \subset \mathbb{M}_B(x_l)$. Pick $\xi = \mu([y', y_m])$. This leads to a contradiction.

Following the same argument, for large enough β's, any $x_h > \tilde{x}$ is not in $\mathbb{M}_S(y_m)$.

Therefore, when β is large enough, $\tilde{x}$ is the unique element in $\mathbb{M}_S(y_m)$. Since the probability of meeting a buyer of type $\tilde{x}$ is zero, the profit of the seller of type y_m is zero. This is a contradiction.

The same proof can be used to show that $\mu(\mathbb{M}_S(y)) \in [0, \xi)$ for any $\xi > 0$.

Step 3: converge to perfect positive (negative) sorting.

Step 3.1: the distance between any two elements in a matching set is arbitrarily small.

First, the monotonicity of value functions implies that the matching set of any buyer or seller is non-empty if there exist at least one lower type with non-empty matching set. In other words, there exists a buyer type x_L, such that any buyer with a type above x_L has a non-empty matching set, while any buyer with a type below x_L has an empty matching set.

Step 2 implies that for any y and $\eta > 0$, there exists an $\epsilon > 0$, such that for any $\beta > 1 - \epsilon$, it must be true that $x_1 - x_2 < \eta$ for any $x_1 > x_2 \in \mathbb{M}_S(y)$. Suppose otherwise, i.e., there exist an η, a y and $x_1, x_2 \in \mathbb{M}_S(y)$ $(x_1 > x_2)$, such that $x_1 - x_2 \geq \eta$ and that $(x_2, x_1) \subsetneq \mathbb{M}_S(y)$. We show that the following claim must be true: the measure of sellers who have any $x \in (x_2, x_1)$ in their matching sets is zero. This contradicts the result of the last paragraph.

We prove the claim by contradiction. First consider a seller of type $y' > y$. If her matching set includes some $x \in (x_2, x_1)$, then based on supermodularity, a neighborhood of y' must be a subset of $\mathbb{M}_B(x_1)$. Suppose that there is a non-empty interval (or several non-empty intervals) of sellers whose matching sets include any buyers in the range. Then the measure of $\mathbb{M}_B(x_1)$ must be strictly bounded above zero. This contradicts the result in step 2. We can use the same approach to prove the claim for sellers with types $y' < y$.

Step 3.2: the existence of $m(x)$ function.

Pick any x, y, y' such that $d(x, y) = 1$ and $y' > y$. We show that under the assumption of supermodularity, there must exist an $x' \in \mathbb{M}_S(y')$ such that $x' \geq x$. Suppose this is not the case. Define $\tilde{x}'$ as $\sup\{\mathbb{M}_S(y')\}$. Then $\tilde{x}' < x$. Using supermodularity, this implies $s(\tilde{x}', y) + s(x, y') > 0$. On the other hand, because $y' > y$, there exists a δ, such that $y' - y > \delta$. Then from the result of step 3.1, $s(\tilde{x}', y) < 0$ and $s(x, y') < 0$. This a a contradiction.

We have established the existence of $m(x)$ function, except for the lower bound of the domain.

Step 3.3: the lower bound of the domain of $m(x)$ converges to $\underline{x}$.

In this step, we show that the lower bound of the domain, denoted as x_L, converges to $\underline{x}$.

We have shown in step 1 that $\frac{f_B(x)}{1-\beta}$ diverges to ∞ for a.e. x. Therefore, $\frac{\bar{\mu}(\mathbb{M}_S)}{1-\beta}$ diverges to ∞, where $\bar{\mu}(\mathbb{M}_S)$ denotes the average trading probability of sellers. Then the stationary condition $1 = M[1 - \beta + \beta\bar{\mu}(\mathbb{M}_S)]$ implies that $M(1 - \beta)$ converges to 0.

Because the matching sets of buyers of type $x < x_L$ are empty, we have

$$F_B(x_L)M(1 - \beta) = \Gamma_B(x_L)$$

If x_L is strictly larger than $\underline{x}$ so that $\Gamma_B(x_L)$ is strictly positive, $F_B(x_L)$ goes to infinity in the limit. This contradicts the fact that $F_B(x_L) \in [0, 1]$.

∎

Proof of Theorem 5

Proof: We only show the proof with supermodular output function $z(x, y)$. The proof when $z(x, y)$ is submodular is essentially the same.

Similar to the proof in step 1 of the proof for Theorem 4, the stationary condition $1 = M[1 - \beta + \beta\bar{\mu}(\mathbb{M}_S)]$ implies that $\frac{f_S(y)}{1-\beta}$ diverges to ∞ for a.e. y. We have also shown that the matching set of any buyer is non-empty in the limit. Therefore, $\frac{1-\beta}{\mu(\mathbb{M}_B(x))} \to 0$ as $\beta \to 1$ for any x. Based on this result, we can show that $V_B'(x) \to z_1(x, m(x))$ whenever $V_B(x)$ is differentiable.

To see why, notice that there exist a $\tilde{y} \in \mathbb{M}_B(x)$ such that

$$V_B'(x) = z_1(x, \tilde{y})\frac{\mu(\mathbb{M}_B(x))}{1 - \beta + \beta\mu(\mathbb{M}_B(x))} \to z_1(x, \tilde{y}) \to z_1(x, m(x))$$

This implies

$$V_B(x) \to V_B(\underline{x}) + \int_{\underline{x}}^{x} z_1(\tilde{x}, m(\tilde{x}))d\tilde{x}$$

$$= V_B(\underline{x}) + \int_{\underline{x}}^{x} dz(\tilde{x}, m(\tilde{x})) - \int_{\underline{y}}^{m(x)} z_2(m^{-1}(\tilde{y}), \tilde{y})d\tilde{y}$$

Since $s(x, m(x)) \to 0$, the price charged by a seller of type $y = m(x)$ can be computed as

$$P(y) \to z(m^{-1}(y), y) - V_B(m^{-1}(y))$$

$$\to z(\underline{x}, \underline{y}) + \int_{\underline{x}}^{x} dz(\tilde{x}, m(\tilde{x})) - [(V_B(\underline{x}) + \int_{\underline{x}}^{x} dz(\tilde{x}, m(\tilde{x})) - \int_{\underline{y}}^{m(x)} z_2(m^{-1}(\tilde{y}), \tilde{y})d\tilde{y}]$$

$$= z(\underline{x}, \underline{y}) - V_B(\underline{x}) + \int_{\underline{y}}^{y} z_2(m^{-1}(\tilde{y}), \tilde{y})d\tilde{y}$$

Here, $V_B(\underline{x}) = 0$ because sellers have all the bargaining power. ∎

Proof of Lemma 5

Proof: The expected time to trade for a buyer of type x (seller of type y) is proportional to $\frac{1-\beta}{\mu(\mathbb{M}_B(x))}$ $(\frac{1-\beta}{\mu(\mathbb{M}_S(y))})$. We know from the proof of Theorem 5 that $\frac{1-\beta}{\mu(\mathbb{M}_B(x))}$ converges to 0 for any x. By the same argument, $\frac{1-\beta}{\mu(\mathbb{M}_S(y))} \to 0$ follows from the fact that $\frac{f_B(x)}{1-\beta}$ diverges to ∞ for a.e. x.

∎

Proof of Lemma 6

Proof: From the steady-state distribution, we know that for any $x > x_L$,

$$\Gamma_B(x) = \Gamma_B(x_L) + \int_{x_L}^{x} [1 - \beta + \beta\mu(\mathbb{M}_B(x))] M f_B(x)dx$$

Because $M(1 - \beta)$ converges to zero, $[1 - \beta + \beta\mu(\mathbb{M}_B(x))] M$ converges to $\mu(\mathbb{M}_B(x))M$. In addition, we know that $\Gamma_B(x_L)$ converges to zero. Therefore,

$$\Gamma_B(x) \to \int_{\underline{x}}^{x} \mu(\mathbb{M}_B(x))M f_B(x)dx$$

Similarly,

$$\Gamma_S(y) \to \int_{\underline{y}}^{y} \mu(\mathbb{M}_S(y))M f_S(y)dy$$

Note that the right-hand sides of the above two equations are the total measure of trade for buyers with types lower than x and for sellers with types lower than y.

If $z(x, y)$ is supermodular, then the total measure of trade for buyers with types lower

than x and for sellers with types lower than $m(x)$ must be the same in the limit. This implies

$$\Gamma_B(x) - \Gamma_S(m(x)) \to 0$$

The case with submodular is similar and the proof is skipped. ∎

Proof of Lemma 7
Proof:

A seller has no incentive to change its price if

$$\frac{d\mu(\mathbb{M}_S(y))}{\mu(\mathbb{M}_S(y))} \leq -\frac{1 - \beta + \beta\mu(\mathbb{M}_S(y))}{1 - \beta} \frac{-dP(y)}{P(y)}$$

Or equivalently,

$$E_y(p) < -\frac{1 - \beta + \beta\mu(\mathbb{M}_S(y))}{1 - \beta}$$

Using the previous results, it is easy to show that the right-hand-side converges to $-\infty$.

Following the same approach, we can show that a buyer has no incentive to change its matching set by varying $V(x)$ if

$$E_x(v) < -\frac{1 - \beta + \beta\mu(\mathbb{M}_B(x))}{1 - \beta}$$

The right-hand-side again converges to $-\infty$ as β converges to one. ∎

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