Inter-Dealer Trades in OTC Markets – Who Buys and Who Sells?

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Abstract

Oftentimes, dealers in an OTC market may not be able to trade with one another whenever they desire to do so, just as investors find it necessary to incur time and effort to buy and sell assets not traded in a centralized exchange. Moreover, an individual dealer can obviously only carry limited quantities of the asset over time and the inventory capacities may certainly differ among dealers. In this environment, dealers trade among themselves, whenever the opportunities arise, to rebalance inventories for facilitating the sale and purchase of the asset to and from investors. The inter-dealer trades naturally resemble a core-periphery trading network documented empirically. Dealers with a smaller capacity occupy peripheral positions in the network and provide inventory for large dealers in the core when inventory is in shortest supply but liquidity when the asset supply is most abundant. The equilibrium is constrained efficient with a competitive inter-dealer market that serves to allocate inventory and spare inventory capacity to dealers who value them the most. The model yields a host of testable implications about how investors' and dealers' trading probabilities, the inter-dealer trading price and its volume depend on the asset supply and the diversity of dealers.

Keywords: OTC Market, Inter-Dealer Trades, Trading Network
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1 Introduction

Many financial assets, including government and corporate bonds, asset-backed securities, and derivatives, are traded in over-the-counter (OTC) markets instead of in centralized exchanges.[1] Two distinguishing features of OTC markets are that trades are almost always intermediated by dealers of various kinds and that the dealers do not just trade with investors but also among themselves. Indeed, inter-dealer trades can account for a significant fraction of the overall transactions for a given asset.[2]

It has long been recognized, going back to Ho and Stoll (1983), that dealers may trade among themselves for inventory risk concerns.[3] In these models, a risk-averse trader having a greater exposure to some risky assets sells a certain fraction of his holding to another risk-averse trader with a lesser initial exposure to the mutual benefits of both. The common understanding seems to be that trading for inventory concerns is inherently linked to risk aversion. But must inter-dealer trades motivated by the sharing of inventory risks necessarily arise from risk aversion?

In many OTC markets, a given dealer cannot take up a certain buy order from an investor unless the dealer possesses a large enough inventory of the asset beforehand if the dealer is not able to acquire the requisite amount of it from other dealers at a short notice. On the other hand, if dealers' inventory capacities are not unbounded and if they cannot sell to other dealers at will, a given dealer can only meet a sell order from an investor if the dealer has sufficient spare inventory capacity at the given moment. Given such constraints, dealers may find it beneficial to trade with one another, whenever they are able to do so, to reach their optimal inventory levels to best prepare themselves for trading with investors. There is good reason to believe that there are inter-dealer trades in an OTC market motivated by the sharing of inventory risk, broadly understood, but not out of any consideration related to risk aversion.

In this paper, we extend the seminal random search models of the OTC market of Duffie, Garleanu and Pedersen (2005) and Lagos and Rocheteau (2009) to study how dealers trade with one another for managing inventory levels for their future trading needs with investors. The point of departure is that, in our model, (1) dealers have only imperfect access to trading with other dealers and (2) they are heterogeneous in their inventory capacities. These are arguably very plausible assumptions. First, to be sure, in reality, there is not a frictionless platform on which dealers can continuously trade among themselves in a typical OTC market, just as investors must expend time and effort in buying and selling the asset. The heterogeneity in inventory capacity among dealers can result from differences in financing costs – dealers who finance asset purchases out of retained earnings and owners' equities can face different opportunity costs of funds, whereas dealers who finance asset purchases by borrowing can be charged different risk premia. The heterogeneity can also be due to risk management considerations or portfolio choices. In this paper, we do not attempt to model how the heterogeneity

[1] As an example, the gross market value of global OTC derivatives totaled 38,286 billion US dollars in 2014 and 29,992 billion US dollars in 2015 (Bank for International Settlement, Semiannual OTC derivatives statistics, updated on May 4, 2016).

[2] Li and Schurhoff (2014) show that in the period covered by their data set, 16 million out of 60 million transactions in municipal bonds are inter-dealer trades. A similar percentage of inter-dealer trade is also documented in Hollifield, Neklyudov and Spatt (2014).

[3] A notable modern variant is in Atkeson, Eisefldt and Weill (2015).

arises endogenously but instead restrict attention to exploring the consequences thereof.

With imperfect access to inter-dealer trading, it becomes imperative for dealers to choose the appropriate levels of inventory holding to be able to meet the uncertain future buy and sell orders from investors. In our model, dealers possessing different inventory capacities attain their respective optimal inventory holdings by buying and selling among themselves when the opportunities come, whereby dealers of different inventory capacities play different roles in the inter-dealer market.

In particular, in our model, there is a given measure of what we call small dealers, each endowed with one unit of inventory capacity, and a given measure of what we call large dealers, each endowed with two units of inventory capacity. At the beginning of each period, investors who value the asset highly but have no asset in hand (high-valuation non-owners hereinafter) and investors who own a unit of the asset but do not value it (low-valuation owners hereinafter) enter the market to buy from and sell to dealers. Investors and dealers randomly meet in this investor-dealer market in which a given dealer can only sell to (buy from) an investor if the dealer has at least one unit of inventory (spare capacity) beforehand. The investor and the dealer in a transaction agree to a price reached via Nash bargaining. All investors who are on the market but fail to trade remain in the market. Once the investor-dealer trades are completed, and only then, a perfectly competitive inter-dealer market opens, through which dealers can rebalance their inventory holdings. Finally, at the end of the period, a fraction of high-valuation owners suffer exogenous liquidity shocks and turn into low-valuation owners, who then enter the investor-dealer market in the next period to sell their assets.

We restrict attention to studying steady-state equilibrium for brevity. We are able to obtain a multitude of analytical results from an apparently very complicated model, in which dealers of the two inventory capacities make decisions for every possible inventory level – decisions which affect and are affected by the steady-state distribution of agents.

Underlying most of the results to follow is a particular ranking of the marginal benefit of inventory among different types of dealers that holds in any steady-state equilibrium with active trading. Specifically, the first unit of inventory is valued higher by a large dealer than by a small dealer since the small dealer, but not the large dealer, would exhaust his entire inventory capacity in acquiring one unit of the asset; by the same token, the last unit of spare inventory capacity is valued higher by a large dealer than by a small dealer since the large dealer, but not the small dealer, need not fill up his entire inventory to already possess a unit of the asset for sale to investors.

The ranking implies that in equilibrium, depending on the asset supply and the extent of dealers' heterogeneity, small dealers either always sell or always buy in the inter-dealer market. Large dealers, on the other hand, sell as well as buy in any equilibrium. If all small dealers only sell or only buy, they do not trade among themselves but only with large dealers. The large dealers, given that they sell as well as buy, trade among themselves, in addition to trading with small dealers. Altogether then, the trading patterns resemble a core-periphery trading network, as documented in Li and Schürhoff (2014) and Hollifield, Neklyudov and Spattand (2014), in which large dealers are in the center, trading among themselves as well as with small dealers in the periphery, who do not trade with one another. Furthermore, the large dealers in the center always hold (weakly) more inventories than small dealers in the periphery do, given the former incur a lower opportunity cost in utilizing their first unit of inventory capacity than the latter do in using up their only unit of inventory capacity, an implication also consistent with the findings in Li and Schürhoff (2014).

If the large core dealers are more interconnected and possess a greater inventory capacity, perhaps it seems intuitive that they should act to provide inventory to the small peripheral dealers. It turns out that the exact opposite holds in our model – it is the small peripheral dealers who provide inventory to the large core dealers. Dealers should need inventory the most in a market with a small asset supply in which the inter-dealer price must rise to dampen the demand for the market to clear, up to the level at which no dealers strictly prefer to buy while all dealers holding a full inventory strictly prefer to sell at least one unit. In what we call the "Selling Equilibrium," small peripheral dealers provide inventory to large core dealers by selling to them. On the other hand, when the asset supply is abundant, so dealers need liquidity the most, the inter-dealer market price must fall to the level at which no dealers strictly prefer to sell while all dealers having an empty inventory strictly prefer to buy at least one unit. In what we call the "Buying Equilibrium," small peripheral dealers provide liquidity to large core dealers by buying from them. In between, what we call the "Balanced Equilibrium" takes hold, in which large dealers holding a full inventory strictly prefer to sell while large dealers with an empty inventory strictly prefer to buy. In the Balanced Equilibrium, small dealers sell to large dealers if the asset supply is relatively meager and buy from large dealers otherwise. Moreover, given the uniqueness of equilibrium, the direction of trade between small and large dealers is persistent for a given set of underlying parameters, another implication of the model that has empirical support with the findings in Li and Schurhoff (2014).

In addition to implications on the structure of trading relationships, our model yields a rich set of other testable implications for future empirical research on the OTC market. We investigate how market-tightness, inter-dealer trading price and volume change with respect to the asset supply and the diversity of dealers. Perhaps somewhat unexpected a priori is that the inter-dealer trading volume is "M-shaped" in response to changes in the asset supply – trading is most active when the asset supply is at a moderately low, but not the lowest, level and at a moderately high, but not the highest, level. Dealers trade among themselves to rebalance inventory, to which the need is greatest when either they find it hardest to acquire inventory or liquidity from investors, i.e., when the asset supply is at the lowest or the highest level. But precisely when the asset supply is at the lowest or the highest level, dealers who possess inventory (spare capacity) to sell (buy) can only be few and far between. In equilibrium, prices must then rise (fall) to dampen the demand (supply). In this way, trading is most active when the demand for and the supply of inventory are both at relatively high levels, arising from there being a moderately high or low asset supply. The inter-dealer trading volume is also non-monotonic with respect to the fraction of large dealers in the dealer population, reaching the maximum level when the fraction is at some intermediate level, whereby the dealer population is most diverse in terms of inventory capacity.

Finally, we show that the equilibrium is constrained efficient. First, in the frictional investor-dealer market, there are gains from trade to follow any and all bilateral meetings between investors and dealers. Without any kind of information imperfection, the mutually beneficial exchanges are guaranteed to take place, with the terms of trade reached by Nash bargaining. Such potentially gainful exchanges are most abundant when the investors on the market meet dealers who possess inventory for sale and dealers who possess spare capacity to buy most frequently. Competition in the inter-dealer market, with which inventories and spare inventory capacities are allocated to dealers who value them the most in equilibrium – the very dealers who have the greatest need for the inventories and spare capacities to facilitate trading with investors – serves to confer investors the most plentiful trading opportunities.

Related Literature

The main differences between this model and the seminal models of OTC markets in Duffie, Garleanu and Pedersen (2005) and Lagos and Rocheteau (2009) are dealers' imperfect access to the inter-dealer market and the heterogeneity of dealers' inventory capacity – two features that make the present model more suitable for studying the inter-dealer trading relationship. In the two aforementioned papers, whenever a dealer trades with an investor, the dealer can instantaneously offset the transaction by trading in a perfectly competitive inter-dealer market that opens at all times. Such an environment, in which a dealer trades with another dealer only if and when he meets an investor, is arguably not the best environment to study inter-dealer trades as the trades have neither persistent direction nor particular structure. Moreover, dealers hold no inventory in this environment as long as they do not value the asset.

A host of recent papers are motivated to explain the empirical finding that the inter-dealer market exhibits a core-periphery structure. For instance, Neklyudov (2015) proposes a random search model assuming that dealers differ in their search abilities. He shows that the dealers with higher search abilities are more interconnected and hence are in the center. However, as the central dealers trade more frequently, their expected inventory levels are also lower in equilibrium, which is inconsistent with the empirical finding. This suggests that other factors, like inventory capacity emphasized in this paper, can also be important factors influencing a dealer's position in inter-dealer trading relationships.

Ours is not the only model of an OTC market in which dealers hold inventory. Lagos, Rocheteau and Weill (2011) demonstrate that dealers hold inventory to speed up future trades when there is a negative shock knocking the market off the steady state, even if dealers do not value the asset. Weill (2011) shows that the same intuition also applies in a competitive dynamic market with a transient selling pressure. Dealers in our model hold inventory also to facilitate trade, as they do not have continuous access to the inter-dealer market. The difference is that they hold inventory even in the steady state and they gain by trading among themselves. Moreover, we can characterize the relationship between a dealer's optimal inventory holding and the dealer's inventory capacity.

Our paper departs from earlier models of inter-dealer trades motivated by inventory risk concerns that follow from Ho and Stoll (1983) by assuming all traders are risk neutral. The dealers in our model are trading to mitigate inventory risks, as those in the earlier models do, in that they trade to eliminate the risks of carrying an insufficient inventory or an insufficient inventory capacity as far as possible for their trading needs with investors.

Why dealers trade among themselves, other than for risk sharing, is a topic of active ongoing research. Colliard and Demange (2015) study post-issuance intermediation chains where each dealer has limited cash endowment, so that they need to trade with one another to amass the cash endowment of a group of dealers. Glode and Opp (2016) argue that when there is an adverse selection problem, a longer intermediation chain can narrow the information gap between two successive dealers and help mitigate the problem of information asymmetry.

The arguments in the two papers work, however, only when the order of transactions among dealers is exogenously fixed in a particular manner. Hugonnier, Lester and Weill (2016) and Shen, Wei and Yan (2015) assume that investors value the asset differently and show that those with intermediate valuations endogenously become dealers as they stay in the market to sell to investors with even higher valuations after buying.[4] While their argument works in an environment where intermediaries value the asset, ours work even if dealers derive no flow payoff from holding the asset.

The rest of the paper is organized as follows. In Section 2, we set up the model and then study the model's equilibrium. In Section 3, we conduct comparative statics analyses and derive implications of the model. The welfare analysis follows in Section 4. Section 5 contains some concluding remarks. All proofs are relegated to the Appendix, which also includes two Propositions on the comparative statics of prices in the investor-dealer market and discussions on how the major results of the paper survive in more general settings.

2 Model and Analysis

2.1 Basic Environment

Time is discrete and runs from $t = 0, 1, 2, ...\infty$. Two groups of agents – investors and dealers – buy and sell an asset with supply fixed at A in an OTC market. A high-valuation investor derives a per period return of $\upsilon > 0$ in holding a unit of the asset, whereas low-valuation investors and dealers derive the same per period return normalized to zero. An individual investor can hold either zero or one unit of the asset at a time and can only buy or sell the asset through dealers of which there are two types: (1) small dealers, each of whom can hold up to one unit of the asset at a time and (2) large dealers, each of whom can hold up to two units.[5] All agents are risk neutral and discount the future at the same factor β.

At the beginning of each period, a measure of e investors enter the market as high-valuation investors with no assets in hand. Together with the entrants in previous periods who have yet to acquire a unit of the asset, they – the high-valuation non-owners – constitute the population of investor-buyers in the market. Among investors who do own a unit of the asset, the low-valuation owners become the investor-sellers in equilibrium.

Each period is divided into two subperiods. In the first subperiod, a decentralized investor-dealer market opens in which the bilateral meetings between investors and dealers take place. We assume that investor-buyers and dealers with at least a unit of the asset for sale (dealer-sellers hereinafter) meet in one segment of the market and investor-sellers and dealers with spare inventory capacity to buy (dealer-buyers hereinafter) meet in another segment of the market. The matches in each market segment are formed in accordance with the same Mortensen-Pissarides constant-returns matching function, whereby, given market tightness $\theta \in [0, \infty)$ for the ratio of the measures of buyers to sellers for the given market segment, a seller meets a buyer

[4] A similar mechanism is proposed in Piazzesi and Schneider (2009) in their analysis of the housing market.

[5] In the Conclusion, we will discuss how our results may extend qualitatively to settings with (1) small and large dealers having greater inventory capacities and (2) more than two types of dealers, each type having different inventory capacities.

at a probability $\eta(\theta) \in [0,1]$, whereas a buyer meets a seller at the probability $\mu(\theta) = \eta(\theta)/\theta$. The meeting probability $\eta(\theta)$ satisfies the usual conditions:

$$\frac{\partial \eta}{\partial \theta} > 0; \quad \frac{\partial^2 \eta}{\partial \theta^2} < 0; \quad \lim_{\theta \to 0} \frac{\partial \eta}{\partial \theta} = 1; \quad \lim_{\theta \to \infty} \frac{\partial \eta}{\partial \theta} = 0.$$

With two market segments, there are two market tightnesses: (1) θ_{ID} for the ratio of the measure of investor-buyers to dealer-sellers and (2) θ_{DI} for the ratio of the measures of dealer-buyers to investor-sellers.

An obvious alternative to our assumed meeting technology is that investors and dealers search and match in one unified market, whereby an investor meets a dealer at some probability $\lambda(\theta)$ and a dealer meets an investor at probability $\kappa(\theta) = \lambda(\theta)/\theta$ for θ denoting the ratio of all dealers to all investors on the market. In this setup, there would be bilateral meetings between two sellers and between two buyers that cannot lead to any profitable exchanges between the agents concerned. In our model, because not all dealers are selling in the investor-dealer market, what is relevant for an investor-buyer's matching probability should be the measure of dealers who are searching to sell but not all dealers who are on the market. Hendershott and Madhavan (2015) report the increasing prevalence of electronic trading platforms for corporate bonds on which investors post their buy and sell orders. A dealer in any of these markets is then usually well informed of whether an investor is buying or selling before he initiates contact with the investor. Assuming that each dealer seeks out all investors, as in the alternative setup, implies that an investor-buyer's matching probability decreases with the measures of dealer-buyers and investor-sellers – agents who are not potentially engaged in the type of matches under consideration. Our assumed meeting technology embodies the standard assumption that how many matches of one type are formed depends only on the measures of agents who may become partners in such matches, whereas the alternative setup assumes that in addition the measures of other types of agents on the market also matter and exert negative effects on the matches of the given type that would be formed. This latter view perhaps is warranted if there exists ample evidence for the aforementioned effects. Assuming the search and matching of investor-buyers and dealer-sellers (investor-sellers and dealer-buyers) is not affected by the measures of investor-sellers and dealer-buyers (investor-buyers and dealer-buyers) on the market has the virtue of choosing a simple versus a complicated setting when there is no compelling reason for choosing the latter.

Prices in the investor-dealer market fall out of the bargaining between the buyers and sellers in the bilateral meetings in which the agents on the two sides are assumed to possess equal bargaining power.[6] An individual dealer may search as both a dealer-buyer and a dealer-seller in the market in a given period but the meeting technology only allows the dealer to meet at most one investor-buyer and one investor-seller in the period.[7] At the end of the subperiod, those high-valuation non-owners who succeed in buying a unit turn into new high-valuation

[6] The assumption of equal bargaining power is without loss of generality and merely serves to simplify.

[7] This simplifying assumption can be understood as a discrete-time version of a continuous-time meeting process. If the time interval between two periods is small enough relative to the arrival rate of a meeting, then the probability of having more than one meeting per period approaches zero. The assumption can also be justified by a dealer's limited execution capacity in reality. We should explain in the Conclusion and in the Appendix how the major results of the analysis survive while relaxing the restriction.

owners, while those low-valuation owners who succeed in selling their units leave the market for good.

In the second subperiod, a competitive inter-dealer market opens, in which dealers buy and sell as many units of the asset among themselves as they see fit at a given market price, subject to their asset holdings and spare inventory capacities. Finally, at the end of the second subperiod, each high-valuation owner, except for those who have just purchased the asset in the current period, turns into a low-valuation owner at a probability $\delta \in (0, 1)$.

A major difference between the present setting and that in the canonical models of Duffie, Garleanu and Pedersen (2005) and Lagos and Rocheteau (2009) is when and how often dealers have access to the competitive inter-dealer market. In the latter models, dealers can continuously access the competitive inter-dealer market. In such an environment, dealers do not hold any inventory at all in the steady state, as they can immediately offset any transaction with investors in the inter-dealer market. In contrast, the dealers in our model access the inter-dealer market only after or before they meet and trade with investors. Without continuous access to trading with other dealers, a dealer in our model can sell to an investor only if the dealer is holding at least a unit of the asset beforehand and thus the dealer may find it optimal not to offload all units of the asset he acquires from investors at the first opportunity. In a similar vein, a dealer may find it optimal not to entirely fill up his inventory in the inter-dealer market, in anticipation of using the spare capacity for trading with an investor-seller if and when he meets one in the next period.

2.2 Value Functions

A small dealer, S_i, $i = 0, 1$, is either holding 0 or 1 unit of the asset at the beginning of a period when the investor-dealer market opens, whereas a large dealer, L_i, $i = 0, 1, 2$, may also be holding up to 2 units of the asset.

In the investor-dealer market, an investor-buyer meets a dealer-seller – any dealer holding at least one unit of the asset in inventory – at probability $\mu(\theta_{ID})$. The dealer-seller can be an S_1, an L_1 or an L_2. Let p_{I_B,S_1}, p_{I_B,L_1}, and p_{I_B,L_2} be the respective prices at which the investor-buyer buys from these different dealers. Then, the investor-buyer has asset value satisfying,

$$\begin{aligned} U^B &= \mu(\theta_{ID}) \left(\beta U_H^{ON} - \frac{n_1^{SD}}{n_S^D} p_{I_B,S_1} - \frac{n_1^{LD}}{n_S^D} p_{I_B,L_1} - \frac{n_2^{LD}}{n_S^D} p_{I_B,L_2} \right) \\ &\quad + (1 - \mu(\theta_{ID})) \beta U^B, \end{aligned} \tag{1}$$

where n_i^{SD} and n_i^{LD} are the respective measures of small and large dealers holding an i-unit inventory,

$$n_S^D = n_1^{SD} + n_1^{LD} + n_2^{LD}$$

the measure of dealer-sellers, and U_H^{ON} the asset value of a high-valuation owner. In defining this value function, we assume that any meetings between an investor-buyer and a dealer-seller all yield a non-negative match surplus. Likewise, we shall assume that any meetings between an investor-seller and a dealer-buyer all yield a non-negative match surplus in defining the value functions in the following. In Lemma 1 below, we show that the assumptions are without loss of generality as they indeed hold in any equilibrium with active trading between investors and dealers.

A high-valuation owner derives a per period return υ from holding a unit of the asset and may turn into a low-valuation owner at probability δ at the end of the period. Hence,

$$U_H^{ON} = \upsilon + \beta\left(\delta U_L^{ON} + (1-\delta)\, U_H^{ON}\right), \tag{2}$$

where U_L^{ON} denotes the asset value of a low-valuation owner who seeks to sell his unit of the asset. In each period, the investor-seller meets a dealer-buyer – any dealer possessing at least one unit of spare inventory capacity – at probability $\eta(\theta_{DI})$. The dealer may be an S_0, an L_0 or an L_1.[8] Let p_{S_0,I_S}, p_{L_0,I_S}, and p_{L_1,I_S} be the respective prices at which the low-valuation investor sells to these different dealers. Hence,

$$\begin{aligned} U_L^{ON} &= \eta(\theta_{DI})\left(\frac{n_0^{SD}}{n_B^D} p_{S_0,I_S} + \frac{n_0^{LD}}{n_B^D} p_{L_0,I_S} + \frac{n_1^{LD}}{n_B^D} p_{L_1,I_S}\right) \\ &\quad + (1-\eta(\theta_{DI}))\, \beta U_L^{ON}, \end{aligned} \tag{3}$$

where

$$n_B^D = n_0^{SD} + n_0^{LD} + n_1^{LD}$$

is the measure of dealer-buyers.

In addition to trading with investors in the investor-dealer market in the first subperiod, dealers may also trade among themselves in the second subperiod in the competitive inter-dealer market. Write V_i^{SD} and W_i^{SD}, $i = 0, 1$, as the respective asset values of a small dealer entering the investor-dealer market in the first subperiod and the inter-dealer market in the second subperiod with an i-unit inventory. If the asset is traded in the inter-dealer market at price p,

$$W_0^{SD} = \max\left\{\beta V_0^{SD}, \beta V_1^{SD} - p\right\}, \tag{4}$$

$$W_1^{SD} = \max\left\{p + \beta V_0^{SD}, \beta V_1^{SD}\right\}, \tag{5}$$

$$V_0^{SD} = \mu(\theta_{DI})\left(W_1^{SD} - p_{S_0,I_S}\right) + (1-\mu(\theta_{DI}))\, W_0^{SD}, \tag{6}$$

$$V_1^{SD} = \eta(\theta_{ID})\left(p_{I_B,S_1} + W_0^{SD}\right) + (1-\eta(\theta_{ID}))\, W_1^{SD}. \tag{7}$$

In (4), an S_0 entering the inter-dealer market chooses between buying a unit in the market and not buying, whereas in (5), an S_1 chooses between selling the unit and not selling. Clearly, if the first dealer strictly prefers to buy where $p < \beta(V_1^{SD} - V_0^{SD})$, the second dealer must strictly prefer not to sell and vice versa. In (6), an S_0 entering the investor-dealer market meets an investor-seller at probability $\mu(\theta_{DI})$ and buys the unit from the investor at price p_{S_0,I_S}. In (7), an S_1 meets an investor-buyer at probability $\eta(\theta_{ID})$ and sells the unit to the investor at price p_{I_B,S_1}.

A large dealer can hold up to two units of the asset in inventory. The asset values, V_i^{LD} and W_i^{LD}, $i = 0, 1, 2$, satisfy, respectively,

$$W_0^{LD} = \max\left\{\beta V_0^{LD}, \beta V_1^{LD} - p, \beta V_2^{LD} - 2p\right\}, \tag{8}$$

[8] In holding a unit in inventory and having one unit of spare inventory capacity, an L_1 is both a dealer-seller and a dealer-buyer in the given period.

$$W_1^{LD} = \max\left\{p + \beta V_0^{LD}, \beta V_1^{LD}, \beta V_2^{LD} - p\right\}, \tag{9}$$

$$W_2^{LD} = \max\left\{2p + \beta V_0^{LD}, p + \beta V_1^{LD}, \beta V_2^{LD}\right\}, \tag{10}$$

$$V_0^{LD} = \mu(\theta_{DI})\left(W_1^{LD} - p_{L_0,I_S}\right) + (1 - \mu(\theta_{DI}))\, W_0^{LD}, \tag{11}$$

$$\begin{aligned} V_1^{LD} \;=\;& \mu(\theta_{DI})(1 - \eta(\theta_{ID}))\left(W_2^{LD} - p_{L_1,I_S}\right) \\ &+ (1 - \mu(\theta_{DI}))\,\eta(\theta_{ID})\left(p_{I_B,L_1} + W_0^{LD}\right) \\ &+ \mu(\theta_{DI})\,\eta(\theta_{ID})\left(p_{I_B,L_1} - p_{L_1,I_S} + W_1^{LD}\right) \\ &+ (1 - \mu(\theta_{DI}))(1 - \eta(\theta_{ID}))\, W_1^{LD}, \end{aligned} \tag{12}$$

$$V_2^{LD} = \eta(\theta_{ID})\left(p_{I_B,L_2} + W_1^{LD}\right) + (1 - \eta(\theta_{ID}))\, W_2^{LD}. \tag{13}$$

2.3 Bargaining

Assuming equal bargaining power, the respective prices an investor-buyer pays to an S_1, an L_1, and an L_2 satisfy,

$$\beta\left(U_H^{ON} - U^B\right) - p_{I_B,S_1} = W_0^{SD} - W_1^{SD} + p_{I_B,S_1}, \tag{14}$$

$$\beta\left(U_H^{ON} - U^B\right) - p_{I_B,L_1} = W_0^{LD} - W_1^{LD} + p_{I_B,L_1}, \tag{15}$$

$$\beta\left(U_H^{ON} - U^B\right) - p_{I_B,L_2} = W_1^{LD} - W_2^{LD} + p_{I_B,L_2}. \tag{16}$$

On the other hand, the respective prices an investor-seller receives from selling to an S_0, an L_0, and an L_1 satisfy,

$$p_{S_0,I_S} - \beta U_L^{ON} = W_1^{SD} - W_0^{SD} - p_{S_0,I_S}, \tag{17}$$

$$p_{L_0,I_S} - \beta U_L^{ON} = W_1^{LD} - W_0^{LD} - p_{L_0,I_S}, \tag{18}$$

$$p_{L_1,I_S} - \beta U_L^{ON} = W_2^{LD} - W_1^{LD} - p_{L_1,I_S}. \tag{19}$$

2.4 Prices and Match Surpluses in the Investor-dealer Market

As we remarked earlier, in defining the value functions in (1)-(3), (6) and (7), and (11)-(13), we assume that there are non-negative match surpluses in any and all meetings in the investor-dealer market. A priori this need not be true as it is not inconceivable that an investor-buyer (investor-seller) may find it optimal to trade with one type of dealer-seller (dealer-buyer) but not others in equilibrium. By Lemma 1 below, the restriction is without loss of generality in any steady-state equilibrium with active trading.

Lemma 1 *In any steady-state equilibrium, the match surplus for meetings between an investor-seller and any dealer-buyer all equals to*

$$z_{I_S} = p - \beta U_L^{ON}, \tag{20}$$

whereby any such exchanges take place at the same price,

$$p_{S_0,I_S} = p_{L_0,I_S} = p_{L_1,I_S} = \frac{p + \beta U_L^{ON}}{2}. \tag{21}$$

The match surplus for meetings between an investor-buyer and any dealer-seller all equals to

$$z_{I_B} = \beta \left(U_H^{ON} - U^B \right) - p, \tag{22}$$

whereby any such exchanges take place at the same price,

$$p_{I_B,S_1} = p_{I_B,L_1} = p_{I_B,L_2} = \frac{p + \beta \left(U_H^{ON} - U^B \right)}{2}. \tag{23}$$

To understand this lemma, consider a trade between an investor-seller and a dealer-buyer holding an i-unit inventory before the trade. If it is optimal for the dealer to exit the inter-dealer market holding just i units of the asset, then the dealer can simply sell the unit he acquires from the investor at price p in the inter-dealer market. Conversely, if it is optimal for the dealer to exit the inter-dealer market holding $i+1$ units of the asset, by acquiring a unit from the investor, the dealer no longer needs to buy in the inter-dealer market, which would have cost him p otherwise. Therefore, for any dealer-buyer, the gain from trade is p minus the price paid to the investor-seller. The key ingredient of this argument is that any dealer can buy and sell the asset at the same price in the competitive inter-dealer market. On the other hand, to the investor-seller, the gain from trade is the receipt of the selling price net of the continuation value of being a low-valuation owner, which is otherwise independent of the identity of the counterparty of the trade. The match surplus, equal to the sum of the surpluses from trade of the two sides, is then the same across all trades between an investor-seller and any dealer-buyer as given in (20). A similar logic explains (22).[9]

2.5 Inter-dealer Market Trades

By (4) and (5), whether a small dealer entering the inter-dealer market wants to buy or sell depends on how the inter-dealer market price p compares with $\beta \left(V_1^{SD} - V_0^{SD} \right)$. Similarly, by (8)-(10), a large dealer entering the market decides to buy or sell by comparing p against $\beta \left(V_1^{LD} - V_0^{LD} \right)$ and $\beta \left(V_2^{LD} - V_1^{LD} \right)$. To proceed, we first establish that:

Proposition 1 $V_1^{LD} - V_0^{LD} \geq V_1^{SD} - V_0^{SD} \geq V_2^{LD} - V_1^{LD}$ *in any active steady-state equilibrium in which* (20) *and* (22) *are non-negative. The first inequality is strict if* (20) *is strictly positive. The second inequality is strict if* (22) *is strictly positive.*

Proposition 1 says that in an active steady-state equilibrium, an L_0 entering the inter-dealer market has the most to gain from acquiring a unit of the asset in the market, followed by an S_0, whereas an L_1 has to the least to gain. Intuitively, $V_1^{LD} - V_0^{LD} \geq V_1^{SD} - V_0^{SD}$ because the opportunity cost for the large dealer in utilizing his first unit of inventory capacity

[9] In general, that any investor-dealer match should yield a non-negative surplus is due to the tendency that there cannot be a greater trade surplus for a given unit of the asset to pass from one dealer to another dealer before it is sold to an investor. See the discussions in the Applendix.

should be lower than the opportunity cost for the small dealer in utilizing his only unit of inventory capacity – in acquiring a unit in the inter-dealer market, the large dealer, but not the small dealer, still has spare inventory capacity to buy one more unit from an investor in the next period to capture any possible surplus of trade. If the latter surplus is strictly positive, then a large dealer gains strictly more from the first unit of inventory than a small dealer does. When acquiring a unit in the inter-dealer market is at the expense of exhausting one's inventory capacity for both the large and small dealers, however, the small dealer should have more to gain than the large dealer $\left(V_1^{SD} - V_0^{SD} \geq V_2^{LD} - V_1^{LD}\right)$ since the large dealer holding a one-unit inventory already has a unit for sale to investors in the upcoming period, whereas the small dealer does not.

The ranking in Proposition 1 implies that L_0s at least weakly prefer to buy and L_2s at least weakly prefer to sell in equilibrium. Who else will buy and sell depends on what price clears the inter-dealer market, a price that must be bounded by

$$p \in \left[\beta\left(V_2^{LD} - V_1^{LD}\right), \beta\left(V_1^{LD} - V_0^{LD}\right)\right].$$

If $p > \beta\left(V_1^{LD} - V_0^{LD}\right)$, all dealers strictly prefer to leave the inter-dealer market with an empty inventory, whereas if $p < \beta\left(V_2^{LD} - V_1^{LD}\right)$, all dealers strictly prefer to leave the market with a filled inventory. The market cannot clear in either case. For any

$$p \in \left(\beta\left(V_2^{LD} - V_1^{LD}\right), \beta\left(V_1^{SD} - V_0^{SD}\right)\right),$$

or

$$p \in \left(\beta\left(V_1^{SD} - V_0^{SD}\right), \beta\left(V_1^{LD} - V_0^{LD}\right)\right),$$

any and all dealers who desire to trade either *strictly* prefer to buy or sell. The market clears only if the parameters conspire to just equate the measures of buyers and sellers. Such a parameter configuration, however, can only make up a zero-measure subset of the parameter space. Equilibrium obtains in general only for p just equal to $\beta\left(V_1^{LD} - V_0^{LD}\right)$, $\beta\left(V_1^{SD} - V_0^{SD}\right)$, or $\beta\left(V_2^{LD} - V_1^{LD}\right)$, at which there is one type of dealer holding a given inventory indifferent between selling and not selling or between buying and not buying. The market may then clear at some particular mixing probability for the mixed strategy played by the marginal buyers or sellers. Furthermore, we can show that:

Lemma 2 *For p equal to $\beta\left(V_1^{LD} - V_0^{LD}\right)$ or $\beta\left(V_1^{SD} - V_0^{SD}\right)$, both* (20) *and* (22) *and p itself are strictly positive, whereas for p equal to $\beta\left(V_2^{LD} - V_1^{LD}\right)$,* (20) *and p itself are equal to zero while* (22) *is strictly positive. In all cases, the candidate equilibria are active equilibria in which the gains from trade between investors and dealers are non-negative.*

Let us first suppose that $p = \beta\left(V_2^{LD} - V_1^{LD}\right)$, in which case an L_1 entering the inter-dealer market feels indifferent between paying p to buy one more unit and not buying. If he does buy, he exhausts his entire inventory capacity and the purchase would be at the expense of giving up the opportunity to buy from an investor in the next period. Meanwhile, given that he already possesses a unit in inventory to begin with, he can sell to an investor in the next period without buying a unit in the inter-dealer market. Thus, the dealer must be worse off acquiring a unit at any positive p. With $p = 0$, a dealer is willing to buy a unit from an investor also only at a zero price, which means that there must be but a zero surplus in a dealer-buyer and investor-seller trade.[10] On the other hand, there would be a positive surplus in a trade between a dealer-seller and an investor-buyer given that the dealer acquires the unit for free while the investor is strictly better off owning a unit than not owning.

In the other two cases, if p were equal to zero, then $V_1^{SD} = V_0^{SD}$ must hold;[11] i.e., a small dealer earns the same expected trade surplus searching as either a buyer or seller in the investor-dealer market. But previously we noted that at $p = 0$, any dealer must be earning just a zero trade surplus as a buyer and a positive trade surplus as a seller in the investor-dealer market. There must then be a strictly positive price in the inter-dealer market.

It is useful to classify equilibrium into three types, corresponding to p equal to each candidate equilibrium price.

The "Selling" Equilibrium In the Selling Equilibrium, $p = \beta\left(V_1^{LD} - V_0^{LD}\right)$. By Proposition 1 and Lemma 2,

$$p = \beta\left(V_1^{LD} - V_0^{LD}\right) > \beta\left(V_1^{SD} - V_0^{SD}\right) > \beta\left(V_2^{LD} - V_1^{LD}\right),$$

from which it follows that no dealers strictly prefer to buy, whereas any dealers, large and small, with a filled inventory strictly prefer to sell. For this reason, we call this the Selling Equilibrium in which the optimal inventory level of a large dealer is zero or one unit, whereas that of a small dealer is zero unit. Even though L_0s are indifferent between buying and not buying, at least a fraction of them must buy in equilibrium since they are the only possible buyers. Without loss of generality, we proceed assuming that L_1s refrain from selling.

The "Balanced" Equilibrium In the Balanced Equilibrium, $p = \beta\left(V_1^{SD} - V_0^{SD}\right)$. By Proposition 1 and Lemma 2,

$$\beta\left(V_1^{LD} - V_0^{LD}\right) > p = \beta\left(V_1^{SD} - V_0^{SD}\right) > \beta\left(V_2^{LD} - V_1^{LD}\right),$$

from which it follows that L_0s strictly prefer to buy one unit while L_2s strictly prefer to sell one unit. We refer to this as the Balanced Equilibrium, in which the optimal inventory level of a large dealer is one unit, whereas that of a small dealer is zero or one unit. For the inter-dealer market to clear, if large dealers selling in the inter-dealer market outnumber large dealers buying in the market, a fraction of S_0s buy; otherwise a fraction of S_1s sell.

The "Buying" Equilibrium In the Buying Equilibrium, $p = \beta\left(V_2^{LD} - V_1^{LD}\right)$. By Proposition 1 and Lemma 2,

$$\beta\left(V_1^{LD} - V_0^{LD}\right) = \beta\left(V_1^{SD} - V_0^{SD}\right) > \beta\left(V_2^{LD} - V_1^{LD}\right) = p = 0,$$

[10] In case investors can trade among themselves, an investor-seller should always be able to sell to an investor-buyer at a positive price, given that the latter gains from holding a unit of the asset. In the present setting, the investor-seller does not have direct access to trading with an investor-buyer but can only trade with a dealer, who may possibly not gain from acquiring the unit in which case the trade can only take place at a zero price.

[11] This is obvious for $p = \beta\left(V_1^{SD} - V_0^{SD}\right)$. Since $V_1^{LD} - V_0^{LD} \geq V_1^{SD} - V_0^{SD}$, the latter must be equal to zero if the former is equal to 0. Then, $p = \beta\left(V_1^{LD} - V_0^{LD}\right) = 0$ must also be followed by $V_1^{SD} = V_0^{SD}$.

from which it follows that no dealers strictly prefer to sell. Meanwhile, any dealers with an empty inventory, large and small, strictly prefer to buy. For this reason, we call this the Buying Equilibrium in which the optimal inventory level of a large dealer is one or two units, whereas that of a small dealer is one unit. For the inter-dealer market to clear, a fraction of L_2s must sell since they are the only possible sellers, while L_1s should refrain from buying.

Discussions

Gains from inter-dealer trades Gains from trade in our model arise (1) out of dealers not having continuous access to trading among themselves and (2) from dealers having different inventory capacities. If dealers do not have continuous access to the inter-dealer market, they adjust their inventories (or equivalently spare inventory capacities) every time the inter-dealer market opens to prepare themselves for future buying or selling opportunities with investors. If dealers possess different inventory capacities, their optimal inventories differ. In this environment, there can be a mutually beneficial trade between a large and a small dealer when one dealer's inventory falls short of while the other dealer's inventory exceeds their respective optimal inventories. There can also be mutually beneficial trades between two large dealers, as the gain to an L_0 from acquiring the first unit of the asset exceeds the loss suffered by an L_2 in forgoing the last unit. On the other hand, there can never be any gain from trade between two small dealers – an S_1 selling to an S_0 merely results in the two dealers switching states.

Identities of buyers and sellers In Table 1, we summarize the optimal inventories of large and small dealers upon exiting and the identities of the buyers and sellers upon entry into the inter-dealer market, where the second line of each cell of the "Buyers" and "Sellers" columns indicate the identities of the marginal buyers and sellers in the three types of equilibrium.

Table 1 shows that large dealers with a one-unit inventory never buy or sell in the inter-dealer market. That is, a one-unit inventory is optimal for a large dealer in any equilibrium. The reason is that a dealer holding a one-unit inventory as well as possessing a unit of spare inventory capacity is able to take advantage of all future trading opportunities with investors given that the dealer meets at most one investor-buyer and one investor-seller per period. Table 1 also shows that at least a fraction of L_0s buy, whereas at least a fraction of L_2s sell in any equilibrium. What differs among the equilibria is the role played by small dealers. In the Selling Equilibrium, S_1s sell while S_0s stay of the market. In the Buying Equilibrium, S_0s buy while S_1s stay out of the market. In the Balanced Equilibrium, small dealers may either sell or buy, depending on whether or not the buyers among large dealers outnumber the sellers.

Equilibrium	Price	Optimum Inventory		Buyers	Sellers
		small dealers	large dealers		
Selling	$p = \beta\left(V_1^{LD} - V_0^{LD}\right)$	0	1 and 0	L_0	L_2, S_1
Balanced	$p = \beta\left(V_1^{SD} - V_0^{SD}\right)$	0 and 1	1	L_0 S_0	L_2 S_1
Buying	$p = \beta\left(V_2^{LD} - V_1^{LD}\right)$	1	1 and 2	S_0, L_0	L_2

Table 1: Prices, Optimal Inventories, Buyers and Sellers in Equilibrium

Core-periphery trading network The trading pattern then resembles a core-periphery trading network, in which large dealers, each of which trades with all dealers, can be thought of as in the core of network, whereas small dealers can be thought of as in the periphery of the network, in that they only trade with large dealers. The prediction is consistent with the empirical findings documented in Li and Schürhoff (2014) and Hollifield, Neklyudov and Spatt (2014).[12]

The persistence of the direction of trade In any type of equilibrium in our model, a large dealer sells to another large dealer at a point in time when the first dealer happens to possess a filled inventory while the other dealer happens to possess an empty inventory. At other times, the two dealers may switch roles when each happens to possess the opposite inventory. In a given type of equilibrium, however, small dealers either always sell to or buy from large dealers. Now, if only one type of equilibrium can hold for a given parameter configuration – a result we will establish in Proposition 2 to follow – the direction of trade between small and large dealers is persistent. The implication is also consistent with the findings in Li and Schürhoff (2014), where it is shown that – given that there is a directional (buy or sell) trade between two dealers in one month, the probability that the same directional trade remains in the next month is 62%.

Difference in inventory Another empirical finding in Li and Schürhoff (2014) is that dealers in the core of the inter-dealer trading network hold more assets in inventory. In all three types of equilibrium in our model, the optimal inventory level for a large dealer (who is in the core of the network) is at least weakly higher than that of a small dealer, given that the opportunity cost in utilizing the first unit of inventory capacity is lower for the large dealer than that for the small dealer in utilizing his only unit of inventory capacity.

Our major results thus far – Proposition 1 and the implications thereof – seemingly rest on a number of questionable assumptions. If small dealers can only hold at most one unit of

[12] Given that the inter-dealer market is assumed a Walrasian market, there is of course no particular prediction as to whom a given dealer is selling to or buying from. Where small dealers who trade in the inter-dealer market are all sellers (buyers), it is by no means far-fetched to say that a trading small dealer cannot be selling to (buying from) another small dealer.

the asset in inventory, they cannot gain by trading among one another by construction. A obvious question to ask is how the core-periphery trading structure may hold if small dealers do gain by trading among one another in case they each possess more than a unit inventory capacity. If large dealers can hold up to two units in inventory and may possess up to two units of spare inventory capacity, perhaps a more natural assumption is that they can meet up to two investor-buyers and two investor-sellers in each period. An important lesson in Li and Schürhoff (2014) and the follow-up study in Henderschott, Li, Livdan and Schürhoff (2015) is that the inter-dealer market is itself a decentralized market as opposed to a Walrasian market. We will discuss how Proposition 1 survives all three generalizations in the Conclusion of the paper, with more details to follow in the Appendix.

A given type of equilibrium places a set of restrictions on the measures of dealer-buyers, dealer-sellers and the asset held by these dealers, which in turn impact on the probabilities at which investors and dealers buy and sell in the investor-dealer market. Then, a candidate equilibrium can indeed be equilibrium only if these restrictions are met and where the probabilities of trades are bounded below one, in addition to the existence of some positive mixing probability for the marginal buyers' or sellers' mixed strategy which clears the inter-dealer market. We now proceed to study the underlying environment as defined by the asset supply A, the turnover rate of high-valuation owners δ, the entry rate of high-valuation non-owners e, and the measures of large and small dealers in which the restrictions of each type of equilibrium are met.

2.6 Accounting Identities, Market Tightness, and Stock-Flow equations

At the beginning of the first subperiod, if the market is populated by n^{SD} small dealers and n^{LD} large dealers,

$$n_0^{SD} + n_1^{SD} = n^{SD}, \tag{24}$$

$$n_0^{LD} + n_1^{LD} + n_2^{LD} = n^{LD}. \tag{25}$$

The asset is in fixed supply equal to A, and hence,

$$n_H^{ON} + n_L^{ON} + n_1^{SD} + n_1^{LD} + 2n_2^{LD} = A, \tag{26}$$

where n_H^{ON} and n_L^{ON}, denote, respectively, the measures of high-valuation owners and low-valuation owners.

Let n_B^I denote the measure of high-valuation non-owners *cum* investor-buyers. Then,

$$\theta_{ID} = \frac{n_B^I}{n_S^D} = \frac{n_B^I}{n_1^{SD} + n_1^{LD} + n_2^{LD}}. \tag{27}$$

Recall that the population of investor-sellers is comprised of the low-valuation owners. Then,

$$\theta_{DI} = \frac{n_B^D}{n_L^{ON}} = \frac{n_0^{SD} + n_0^{LD} + n_1^{LD}}{n_L^{ON}}. \tag{28}$$

In the steady state, the respective inflows and outflows of high-valuation owners, low-valuation owners, and investor-buyers are equal. Hence,

$$n_B^I \mu(\theta_{ID}) = \delta n_H^{ON}, \tag{29}$$

$$\delta n_H^{ON} = \eta(\theta_{DI}) n_L^{ON}, \tag{30}$$

$$e = n_B^I \mu(\theta_{ID}). \tag{31}$$

Not all n_i^{SD} and n_i^{LD} can be positive in a given type of equilibrium. In the Selling Equilibrium for example, by the third and fourth columns of Table 1, all small dealers exit the inter-dealer market with an empty inventory, whereas large dealers may do so with either an empty or a one-unit inventory. The measures of small and large dealers with various levels of inventory when the investor-dealer market opens are as depicted in Table 2.

	Selling Equilibrium	Balanced Equilibrium	Buying Equilibrium
n_0^{SD}	n^{SD}	$[0, n^{SD}]$	0
n_1^{SD}	0	$[0, n^{SD}]$	n^{SD}
n_0^{LD}	$[0, n^{LD}]$	0	0
n_1^{LD}	$[0, n^{LD}]$	n^{LD}	$[0, n^{LD}]$
n_2^{LD}	0	0	$[0, n^{LD}]$

Table 2: Measures of dealers entering the investor-dealer market in the three types of equilibrium

Given the measures of dealers, n_i^{SD}, $i = 0, 1$, and n_i^{LD}, $i = 0, 1, 2$, when the investor-dealer market opens in the first subperiod, the corresponding measures of dealers leaving the market and entering the inter-dealer market in the second subperiod, denoted as m_i^{SD}, $i = 0, 1$, and m_i^{LD}, $i = 0, 1, 2$, are given by the following.

$$m_0^{SD} = (1 - \mu(\theta_{DI})) n_0^{SD} + \eta(\theta_{ID}) n_1^{SD}, \tag{32}$$

$$m_1^{SD} = \mu(\theta_{DI}) n_0^{SD} + (1 - \eta(\theta_{ID})) n_1^{SD}, \tag{33}$$

$$m_0^{LD} = (1 - \mu(\theta_{DI})) \left(n_0^{LD} + \eta(\theta_{ID}) n_1^{LD}\right), \tag{34}$$

$$\begin{aligned} m_1^{LD} &= \mu(\theta_{DI}) n_0^{LD} + [\mu(\theta_{DI}) \eta(\theta_{ID}) + (1 - \mu(\theta_{DI}))(1 - \eta(\theta_{ID}))] n_1^{LD} \\ &\quad + \eta(\theta_{ID}) n_2^{LD}, \end{aligned} \tag{35}$$

$$m_2^{LD} = (1 - \eta(\theta_{ID})) \left(\mu(\theta_{DI}) n_1^{LD} + n_2^{LD}\right). \tag{36}$$

For example, (32) says that S_0s entering the inter-dealer market are among the S_0s entering the investor-dealer market who fail to buy a unit in the market and the S_1s who succeed in selling the unit.

2.7 Market clearing in the Inter-dealer Market

Selling Equilibrium In the Selling Equilibrium, all dealers entering the inter-dealer market with a filled inventory strictly prefer to sell, whereas the only dealers who weakly prefer to buy are L_0s. For the inter-dealer market to clear, the measure of dealers who strictly prefer to sell must not exceed the measure of dealers who weakly prefer to buy; i.e.,

$$m_1^{SD} + m_2^{LD} \leq m_0^{LD}. \tag{37}$$

Balanced Equilibrium In the Balanced Equilibrium, L_0s entering the inter-dealer market strictly prefer to buy, whereas L_2s strictly prefer to sell. Then, if the latter outnumber the former; i.e.,

$$m_2^{LD} \geq m_0^{LD}, \tag{38}$$

a fraction of S_0s entering the inter-dealer market must buy in equilibrium; otherwise, a fraction of S_1s must sell in equilibrium. Hence, in case (38) holds, the inter-dealer market clears if

$$m_2^{LD} - m_0^{LD} \leq m_0^{SD}; \tag{39}$$

otherwise the inter-dealer market clears if

$$m_0^{LD} - m_2^{LD} \leq m_1^{SD}. \tag{40}$$

Buying Equilibrium In the Buying Equilibrium, all dealers entering the inter-dealer market with an empty inventory strictly prefer to buy, whereas the only dealers who weakly prefer to sell are L_2s. For the inter-dealer market to clear, the measure of dealers who strictly prefer to buy must not exceed the measure of dealers who weakly prefer to sell; i.e.,

$$m_0^{SD} + m_0^{LD} \leq m_2^{LD}. \tag{41}$$

2.8 Equilibrium

Given $\{n^{SD}, n^{LD}, A, e, \delta\}$, a steady-state equilibrium consists of the respective non-negative values of n_0^{SD}, n_1^{SD}, n_0^{LD}, n_1^{LD}, n_2^{LD}, n_H^{ON}, n_L^{ON} and n_B^{I} that satisfy (24)-(31), the restrictions on n_i^{SD} and n_i^{LD} in Table 2 and the market-clearing conditions for the type of equilibrium under consideration in (37)-(41). Write $n^D = n^{SD} + n^{LD}$ as the total measure of dealers.

Proposition 2 *The Selling Equilibrium and the Buying Equilibrium may only hold for $e < n^{LD}$. The Balanced Equilibrium may only hold for $e < n^{LD} + \frac{n^{SD}}{2}$.*

a. For $e < n^{LD}$, define

$$B_S \equiv e + \frac{n^D}{\mu^{-1}\left(\frac{e}{n^D}\right)},$$

$$B_M \equiv n^{LD} + \frac{n^D}{\mu^{-1}\left(\frac{e}{n^D}\right)},$$

$$B_L \equiv n^D + \frac{n^{LD}}{\mu^{-1}\left(\frac{e}{n^{LD}}\right)},$$

where $B_S \leq B_M \leq B_L$.

(i) for $A - e/\delta \in (B_S, B_M]$*, the Selling Equilibrium holds,*

(ii) for $A - e/\delta \in [B_M, B_L]$*, the Balanced Equilibrium holds,*

(iii) for $A - e/\delta \geq B_L$*, the Buying Equilibrium holds.*

b. *For* $e \in \left[n^{LD}, n^{LD} + \frac{n^{SD}}{2}\right)$*, the Balanced Equilibrium exists if*

$$A - \frac{e}{\delta} > e + \frac{n^D + n^{LD} - e}{\mu^{-1}\left(\frac{e}{n^D + n^{LD} - e}\right)} \equiv \mathcal{B}_M.$$

c. *In the Balanced Equilibrium,*

(i) for $A - e/\delta < S$*, small dealers sell in equilibrium,*

(ii) for $A - e/\delta > S$*, small dealers buy in equilibrium,*

(iii) for $A - e/\delta = S$*, small dealers do not trade in the inter-dealer market, where*

$$S \equiv n^{LD} + \frac{n^{SD}}{2} + \frac{n^{LD} + \frac{n^{SD}}{2}}{\mu^{-1}\left(\frac{e}{n^{LD} + \frac{n^{SD}}{2}}\right)}.$$

For $e < n^{LD}$*,* $B_M \leq S \leq B_L$*, whereas for* $e \in \left[n^{LD}, n^{LD} + \frac{n^{SD}}{2}\right)$*,* $\mathcal{B}_M \leq S$*.*

A steady-state equilibrium exists only if the necessary conditions on e, the entry rate of investors, stated at the beginning of the Proposition, are met. These conditions arise because if a measure of e investors enter the market as high-valuation non-owners in each period, in the steady state, there have to be the same measure of e investors exiting high-valuation non-ownership after buying a unit and the same measure of e investors exiting low-valuation ownership and the market altogether after selling their units in the same period. All this requires that more than e dealer-sellers and more than e dealer-buyers are present in the market. In the Selling Equilibrium, only large dealers sell in the investor-dealer market, whereas in the Buying Equilibrium, only large dealers buy in the investor-dealer market. Then, for either type of equilibrium to exist, a necessary condition is that $e < n^{LD}$. In the Balanced Equilibrium, however, a fraction of small dealers enter the investor-dealer market with an empty inventory and a fraction enter with a filled inventory. Then, there will also be small dealers among both dealer-buyers and dealer-sellers and a steady-state equilibrium may exist even for $e \geq n^{LD}$ but not for $e \geq n^{LD} + \frac{n^{SD}}{2}$ since if more than one-half of all small dealers search as buyers (sellers), then there can only be fewer than one-half searching as sellers (buyers) in the investor-dealer market.

The conditions in Parts (a)-(c) of the Proposition can be interpreted as conditions on the inventory of the asset held by dealers,

$$A^D \equiv A - n_H^{ON} - n_L^{ON} = A - \frac{e}{\delta} - \frac{n_B^D}{\mu^{-1}\left(\frac{e}{n_B^D}\right)},$$

for each type of equilibrium to hold.[13] In a given type of equilibrium, A^D is first of all bounded by the measures of dealers who may be holding inventory of the asset. For example, in the Selling Equilibrium, small dealers do not hold any inventory while a fraction of large dealers may each hold a unit, in which case[14]

$$0 \leq A^D \leq n^{LD}.$$

Moreover, in the steady state, on the one hand, dealers' asset holding must be sufficiently plentiful for them to sell e units to the high-valuation non-owners. On the other hand, it must not exceed the level that would leave the dealers with insufficient spare inventory capacity to buy e units from the low-valuation owners. All together, A^D must be bounded by[15]

$$e < A^D < n^{LD} + n^D - e. \tag{42}$$

It can then be shown that combining the two requirements yield the conditions in Parts (a)-(c).[16,17]

Among the three types of equilibrium, the Selling Equilibrium, in which only a fraction of large dealers may hold just a one-unit inventory, involves dealers holding the least inventory, whereas the Buying Equilibrium, in which only a fraction of large dealers may still have one unit of spare inventory capacity, involves dealers holding the largest inventory. Part a(i) of the Proposition can be interpreted to say that when A^D, held entirely by large dealers, just suffices to satisfy the demand from investor-buyers, the Selling Equilibrium begins to hold and it holds until A^D is up to the level at which all large dealers are holding a unit. At this point, according to Part a(ii), the Balanced Equilibrium begins to hold and it holds until A^D is up to the level at which all small dealers are holding a unit in inventory as well. Thereafter, by Part a(iii), the Buying Equilibrium holds, in which all dealers hold at least a one-unit inventory and a fraction of large dealers are holding a two-unit inventory. Part (b) of the Proposition, as in Part (a), says that a steady-state equilibrium exists once A^D is up to the level to satisfy investors' asset demand.

[13] In each period in the steady state, a measure of e investors enter high-valuation ownership, whereas each exits at the rate δ, from which $n_H^{ON} = e/\delta$ follows. Meanwhile, with the measure of low-valuation owners selling in each period equal to e, $\eta(\theta_{DI})\, n_L^{ON} = e$, from which $n_L^{ON} = n_B^D / \mu^{-1}\left(\frac{e}{n_B^D}\right)$ obtains given $\theta_{DI} = n_B^D / n_L^{ON}$.

[14] In the Balanced Equilibrium, all large dealers and a fraction of small dealers hold a one-unit inventory, in which case $A^D \in \left[n^{LD}, n^D\right]$. In the Buying Equilibrium, all small dealers and a fraction of large dealers hold a one-unit inventory with the rest of the large dealers holding a two-unit inventory, in which case $A^D \in \left[n^D, n^D + n^{LD}\right]$.

[15] A^D cannot be just equal to but must exceed e, for if $A^D = e$, there would have to be an arbitrarily large n_B^I to cause $\theta_{ID} = n_B^I / n_S^D \to \infty$ for each dealer-seller to sell at probability one. Likewise, A^D cannot be just equal to but must fall below $n^{LD} + n^D - e$, for if $A^D = n^{LD} + n^D - e$, there would have to be an arbitrarily large n_L^{ON} to cause $\theta_{DI} = n_B^D / n_L^{ON} = 0$ for each dealer-buyer to buy at probability one.

[16] In the Selling Equilibrium, the two requirements combine into $e < A^D \leq n^{LD}$. In the Balanced Equilibrium, the two requirements combine into $n^{LD} \leq A^D \leq n^D$ for $e < n^{LD}$ and $e < A^D < n^D + n^{LD} - e$ for $e \in \left[n^{LD}, n^{LD} + \frac{n^{SD}}{2}\right)$. In the Buying Equilibrium, the two requirements combine into $n^D \leq A^D < n^{LD} + n^D - e$.

[17] To the extent that A^D depends on n_B^D, which differs across the equilibria, A^D differs across the equilibria too. In the Selling Equilibrium, $n_B^D = n^D$. In the Balanced Equilibrium $n_B^D = n^D$ ($n_B^D = n^{LD}$) when $A^D = n^{LD}$ ($A^D = n^D$) for $e < n^{LD}$ and $n_B^D = n^{LD} + n^D - e$ ($n_B^D = e$) when $A^D = e$ ($A^D = n^{LD} + n^D - e$) for $e \in \left[n^{LD}, n^{LD} + \frac{n^{SD}}{2}\right)$. In the Buying Equilibrium, $n_B^D = n^{LD}$ ($n_B^D = e$) when $A^D = n^D$ ($A^D = n^{LD} + n^D - e$).

Parts a(iii) and (b) of the Proposition say that a steady-state equilibrium holds even for arbitrarily large $A - e/\delta$, which can be interpreted as how the upper bound on A^D in (42) will never be reached. In our model, as the asset supply increases and the sellers' side of the investor-dealer market is becoming increasingly congested, it takes longer and longer for each low-valuation owner to sell, during which the measure of low-valuation owners and their asset holdings increase without bounds. With n_L^{ON} increasing in tandem with the asset supply, the inventory held by dealers never rises above the level that would leave them with insufficient spare inventory capacity to buy e units of the asset from investors.

In Part (c) of the Proposition, when $A - e/\delta = S$ just holds in the Balanced Equilibrium, A^D is at the level at which dealers buy from and sell to investors at the same probability. Then, there would be just as many L_0s and L_2s entering the inter-dealer market, in which case small dealers do not trade in the market. For any smaller (larger) A^D, dealers buy at a smaller (larger) probability while they sell at a higher (smaller) probability in the investor-dealer market to result in fewer (more) L_2 dealers entering the inter-dealer market to sell than L_0 dealers entering the market to buy. Small dealers sell (buy) in equilibrium to eliminate the excess demand (supply) among large dealers.

The proof of the Proposition in the Appendix shows that the condition in Parts (a)-(c) are also the conditions for how the inter-dealer market can clear for the given type of equilibrium in (37)-(41), which requires that there are fewer dealers who strictly prefer to trade in one direction than there are dealers who weakly prefer to trade in the opposite direction. The reason is as follows. While all dealers at least weakly prefer to reverse the transactions with investors by trading in the inter-dealer market, those who strictly prefer to trade are those who need to regain their respective optimal inventories after trading with investors. These dealers must only be a subset of all dealers who sell to (buy from) investors given that there are two optimal inventory levels for a given type of dealers in each type of equilibrium.[18] Now, in the steady state, there must be the same measure of dealers selling to and buying from investors in each period and thus it follows that dealers who weakly prefer to reverse the transactions with investors in one direction must be more numerous than those who strictly prefer to do so in the other direction. Thus, if the equilibrium conditions for the investor-dealer market of a given type of equilibrium in Parts (a)-(c) hold, the market-clearing condition for the inter-dealer market is guaranteed to be satisfied.

3 Comparative Statics

By Proposition 2, given existence, the equilibrium is unique. The model then yields unambiguous comparative statics results with respect to each underlying parameter.

[18] A dealer who enters the investor-dealer market with an i-unit inventory and acquires one more unit from an investor does not strictly prefer to sell in the inter-dealer market afterwards if the dealer is indifferent between holding an i- and an $(i + 1)$-unit inventory; contrariwise, a dealer who enters the investor-dealer market with an i-unit inventory and sells one unit to an investor does not strictly prefer to buy in the inter-dealer market if the dealer is indifferent between holding an i- and an $(i - 1)$-unit inventory.

3.1 Asset Supply[19]

The asset supply, which may vary with the borrower's changing needs for funds, for instance, by Proposition 2, plays a major role in shaping the equilibrium outcomes. In this section, we look more closely into how changing asset supply affects the direction of trade between large and small dealers, which type of equilibrium holds, how quickly investors and dealers buy and sell, and the inter-dealer trading price and volume.

Who sells to whom? In other network-theoretic models of inter-dealer trades with a core-periphery structure, the core dealers are either identified with dealers that sell to and thus provide inventory for peripheral dealers[20] or dealers that simply tend to trade more frequently.[21] Our model has more specific and arguably more subtle implications on the direction of trade between core and peripheral dealers.

To begin, a direct corollary of Proposition 2 is that large dealers (holding a two-unit inventory) sell to small dealers (with an empty inventory) for $A > S + e/\delta$, whereas for $A < S + e/\delta$, large dealers (with an empty inventory) buy from small dealers instead.[22] In our model then, if the large dealers, who are in the core, are to be interpreted as providing inventory (liquidity) for small dealers, they do so when the asset supply is relatively abundant (meager), just when small dealers should need inventory (liquidity) the least.

Dealers need inventory more than spare inventory capacity in a market with a small asset supply – the scarcity of the asset should give rise to relatively more selling opportunities than buying opportunities for dealers in the investor-dealer market.[23] The competitive inter-dealer market should then serve to allocate the inventory to the dealers who value them (L_0s) more over those who value them less (S_0s), whereby S_1s sell to L_0s. On the other hand, dealers need spare inventory capacity more than inventory in a market with a large asset supply – the abundance of the asset should give rise to relatively more buying opportunities than selling opportunities for dealers in the investor-dealer market. The competitive inter-dealer market should then serve to allocate the spare capacity to dealers who value them more (L_2s) over those who value them less (S_1s), whereby S_0s buy from L_2s. All together, a more natural interpretation in our model is that it is the small peripheral dealers that provide inventory (liquidity) for the large core dealers when the latter need inventory (liquidity) the most.

[19] It is straightforward to verify that an increase in δ has the same qualitative effects on the endogenous variables as those of an increase in A. In the steady state, the measure of assets held by high-valuation owners and therefore not in circulation is e/δ, which declines with δ. Then, where there is a larger δ, effectively, more of the asset is on the market for trading among dealers and investors, just like when there is a larger asset supply.

[20] In Farboodi (2014), large banks in the core initiate risky investment projects and acquire funding from small banks in the periphery. In Zhong (2014), the core dealer acquires a risky asset from an investor and sells to other dealers to which the dealer is connected for risk-sharing.

[21] For example, Neklyudov (2015) and Hugonniery et al. (2016).

[22] For $e < n^{LD}$, in both the Selling Equilibrium, which holds for $A - e/\delta \in (B_S, B_M]$, and in the Balanced Equilibrium for $A - e/\delta \in \in [B_M, S)$, small dealers sell to large dealers. In both the Balanced Equilibrium for $A - e/\delta \in \in (S, B_L]$ and the Buying Equilibrium, which holds for $A - e/\delta \geq B_L$, small dealers buy from large dealers. For $e \in \left[n^{LD}, n^{LD} + \frac{n^{SD}}{2}\right)$, only the Balanced Equilibrium can hold, in which the cutoff value for $A - e/\delta$ is similarly S.

[23] That is, it should be easier for dealers to meet investor-buyers than investor-sellers. This is established in Proposition 3a below.

Market Tightness and Turnover In the above, we remarked that dealers should find it easier to buy but more difficult to sell in a market with more abundant asset supply. We state the formal results in the following proposition.

Proposition 3a *(i) For $e < n^{LD}$, as A increases from $B_L + e/\delta$ at which the Selling Equilibrium first holds, $\partial\theta_{DI}/\partial A = 0$ and $\partial\theta_{ID}/\partial A < 0$. Once A reaches $B_M + e/\delta$ at which the Balanced Equilibrium begins to hold, $\partial\theta_{DI}/\partial A < 0$ and $\partial\theta_{ID}/\partial A < 0$. Finally, when A rises up to and above $B_L + e/\delta$ at which the Buying Equilibrium holds, $\partial\theta_{DI}/\partial A < 0$ and $\partial\theta_{ID}/\partial A = 0$. In the transition from one equilibrium type to another, θ_{DI} and θ_{ID} are continuous. (ii) For $e \in \left[n^{LD}, n^{LD} + \frac{n^{SD}}{2}\right)$ and that $A > \mathcal{B}_M + e/\delta$ at which the Balanced Equilibrium holds, $\partial\theta_{DI}/\partial A < 0$ and $\partial\theta_{ID}/\partial A < 0$.*

Proposition 3a implies that indeed, in the investor-dealer market, the dealer-buyer's matching rate $\mu(\theta_{DI})$ is (weakly) increasing and the dealer-seller's matching rate $\eta(\theta_{ID})$ is (weakly) decreasing in A. Perhaps somewhat unexpected a priori is that θ_{DI} in the Selling Equilibrium and θ_{ID} in the Buying Equilibrium do not vary with A. In the Selling Equilibrium, given that all dealers enter the investor-dealer market with at least one unit of spare inventory capacity, all dealers are dealer-buyers in which case n_B^D remains fixed at n^D throughout. To follow is the same θ_{DI} in the steady state for all admissible values of A for otherwise, the measure of assets bought by dealers $(n^D\mu(\theta_{DI}))$ cannot remain equal to e. That θ_{ID} in the Buying Equilibrium does not vary with A can be explained similarly.

Inter-dealer Trading Prices Proposition 3a shows that dealers find it easier to buy from and harder to sell to investors as A increases. If dealers who have bought from investors tend to sell and dealers who have sold to investors tend to buy afterwards in the inter-dealer market, an increase in A should be followed by an increase in supply and a decline in demand in the inter-dealer market and a concomitant decline in the inter-dealer market price. Besides, when one equilibrium type turns into another, the inter-dealer market price p changes its anchor from one indifference condition to another. As such, a minute change in the asset supply can cause a catastrophic change in p. The next proposition formally states the results.

Proposition 3b *(i) For $e < n^{LD}$, as A increases from $B_L + e/\delta$ at which the Selling Equilibrium first holds, p is continuously decreasing in A. Once A reaches $B_M + e/\delta$ at which the Balanced Equilibrium begins to hold, there will be a discrete fall in p, followed by further continuous decreases as A increases further. Finally, when A rises up to and above $B_L + e/\delta$ at which the Buying Equilibrium holds, there will be another discrete fall in p all the way to zero. (ii) For $e \in \left[n^{LD}, n^{LD} + \frac{n^{SD}}{2}\right)$ and that $A > \mathcal{B}_M + e/\delta$ at which the Balanced Equilibrium begins to hold, p is continuously decreasing in A throughout.*

Inter-dealer Trading Volume In the inter-dealer market, trades are driven by the infra-marginal buyers' or sellers' desire to rebalance inventories. The trading volume (TV) in the Selling, Balanced, and the Buying Equilibria are then given by, respectively,

$$TV = m_1^{SD} + m_2^{LD},$$

$$TV = \begin{cases} m_0^{LD} & A \leq S + \frac{e}{\delta} \\ m_2^{LD} & A \geq S + \frac{e}{\delta} \end{cases},$$

$$TV = m_0^{SD} + m_0^{LD}.$$

Proposition 3c *The inter-dealer market trading volume changes non-monotonically with A, as depicted in the table below.*

Selling Equilibrium	Balanced Equilibrium		Buying Equilibrium
	$A \leq S + \frac{e}{\delta}$ small dealers sell	$A \geq S + \frac{e}{\delta}$ small dealers buy	
$\frac{\partial TV}{\partial A} > 0$	$\frac{\partial TV}{\partial A} < 0$	$\frac{\partial TV}{\partial A} > 0$	$\frac{\partial TV}{\partial A} < 0$

For $e < n^{LD}$, the trading volume changes continuously as one equilibrium type changes to another, peaking at $TV = e\left(1 - \frac{e}{n^D}\right)$, when the Selling Equilibrium turns into the Balanced Equilibrium and when the Balanced Equilibrium turns into the Buying Equilibrium.

The comparative statics in Proposition 3c follow from the tendency that as A increases, it becomes easier for dealers to buy and harder for them to sell in the investor-dealer market as shown in Proposition 3a. In the Selling Equilibrium and Balanced Equilibrium where small dealers buy, sales in the inter-dealer market are driven by dealers selling the inventories they acquire from investors, who become more numerous as more dealers manage to buy from investors to follow a given increase in A. In the Buying Equilibrium and Balanced Equilibrium where dealers sell, sales in the inter-dealer market are driven by dealers refurbishing the inventories they sell to investors, who become less numerous as fewer dealers manage to sell to investors to follow a given increase in A.

The most interesting aspect of the Proposition is that it implies that the inter-dealer market is most active when the asset supply is at a relatively low level but not at the lowest level or at a relatively high level but not at the highest level. In our model, with a low (high) level of asset supply, only few dealers are able to buy from (sell to) investors, and those that fail to do so may wish to buy (sell) in the inter-dealer market to rebalance inventories. But if the asset supply is at the lowest (highest) levels, dealers who have units of the asset for sale (spare inventory capacity to buy) in the market can only be few and far between. Then, where the volume of trade in the market is determined by the measure of dealers on the short side of the market, trades are most plentiful just when the asset supply is at a moderately low or at a moderately high level.

3.2 Measure of Large Dealers n^{LD}

A major point of departure in the present model from previous models is that dealers are heterogenous in inventory capacity. In this section, we study how that heterogeneity affects the equilibrium outcomes. In the comparative statics exercises below, we hold constant the total measure of dealers and vary the measure of large dealers. To the extent that there is least diversity among dealers when all dealers are small dealers and when all dealers are large dealers, the comparative statics with respect to n^{LD} we present below can also be interpreted as the impacts of the diversity of dealers in the dealer population on equilibrium outcomes.

Types of Equilibrium If there were no large dealers, any inter-dealer trades would only be between an S_1 selling to an S_0. The equilibrium terms of trade must then be such that the two parties are indifferent between trading and not trading; i.e. $p = \beta\left(V_1^{SD} - V_0^{SD}\right)$ as in the Balanced Equilibrium.[24] For any positive n^{LD} not up to e, by Proposition 2, there can still only be a Balanced Equilibrium in which small dealers remain indifferent between trading and not trading. But as n^{LD} rises up to and above e, the Selling and the Buying Equilibrium may begin to hold in which small dealers strictly prefer to trade. The next proposition traces the evolution of equilibrium type as n^{LD} increases from the smallest admissible value to n^D.

Proposition 4a *Holding fixed n^D, the Balanced Equilibrium begins to hold for*

$$n^{LD} > \eta^{-1}\left(\frac{e}{A - \frac{e}{\delta} - e}\right)\left(A - \frac{e}{\delta} - e\right) - n^D + e.$$

As n^{LD} increases from the lower bound in the above towards n^D, the Balanced Equilibrium holds throughout only if $A - e/\delta = \overline{B}$ holds exactly, where

$$\overline{B} \equiv n^D + \frac{n^D}{\mu^{-1}\left(\frac{e}{n^D}\right)}.$$

In general, the Balanced Equilibrium changes into the Buying Equilibrium for $A - \frac{e}{\delta} > \overline{B}$ at

$$n^{LD} = \eta^{-1}\left(\frac{e}{A - \frac{e}{\delta} - n^D}\right)\left(A - \frac{e}{\delta} - n^D\right),$$

but into the Selling Equilibrium for $A - \frac{e}{\delta} < \overline{B}$ at

$$n^{LD} = A - \frac{e}{\delta} - \frac{n^D}{\mu^{-1}\left(\frac{e}{n^D}\right)}.$$

The Proposition says that as large dealers rise in number with a one-for-one decline in the number of small dealers, the Balanced Equilibrium in general must give way to either the Buying or the Selling Equilibrium. In particular, if the asset supply is relatively abundant with $A - e/\delta > \overline{B}$, for the inter-dealer market to clear in the Balanced Equilibrium, small dealers should buy where there tend to be more L_2s entering the inter-dealer market than L_0s. When n^{LD} increases and n^{SD} falls down to some given levels, the remaining S_0s would no longer suffice to fill the gap between the demand and supply from large dealers. Equilibrium, then, can only obtain when L_2s no longer strictly prefer to sell as when the inter-dealer market price falls from $p = \beta\left(V_1^{SD} - V_0^{SD}\right)$ to $p = \beta\left(V_2^{LD} - V_1^{LD}\right)$, at which point the Buying Equilibrium takes hold. A similar reasoning explains the transition from the Balanced Equilibrium into the Selling Equilibrium when the asset supply is relatively meager with $A - e/\delta < \overline{B}$.

[24] Of course, $n^D = n^{SD}$ has to be large enough to sustain a steady-state equilibrium. By the first condition of Proposition 4a, the condition is $n^D > \eta^{-1}\left(\frac{e}{A-\frac{e}{\delta}-e}\right)\left(A - \frac{e}{\delta} - e\right) + e$.

Market Tightness and Turnover If more of the dealers are large dealers possessing a two-unit inventory capacity, there will be a greater overall inventory capacity among dealers. Then, first of all, there will tend to be more dealer-buyers. Furthermore, when more dealers are buying from investors, dealers' overall inventory holding tends to increase as well, giving rise to there being more dealer-sellers.

Proposition 4b *Holding fixed n^D, as n^{LD} increases from the smallest admissible value for which the Balanced Equilibrium holds, $\partial\theta_{DI}/\partial n^{LD} > 0$ and $\partial\theta_{ID}/\partial n^{LD} < 0$. If and when the Balanced Equilibrium gives way to the Buying Equilibrium, $\partial\theta_{DI}/\partial n^{LD} > 0$ and $\partial\theta_{ID}/\partial n^{LD} = 0$. On the other hand, if and when the Balanced Equilibrium gives way to the Selling Equilibrium, $\partial\theta_{DI}/\partial n^{LD} = 0$ and $\partial\theta_{ID}/\partial n^{LD} = 0$. The two market tightnesses are continuous at the point at which the Balanced Equilibrium turns into either the Buying or the Selling Equilibrium.*

The substantive implication of Proposition 4b is that the investor-sellers' matching rate $\eta(\theta_{DI})$ and the investor-buyers' matching rate $\mu(\theta_{ID})$ are both weakly increasing in n^{LD}. In this way, a market with relatively more large dealers functions strictly better at transferring units of the asset from low- to high-valuation investors, except when it is in the Selling Equilibrium. In the Selling Equilibrium, in which all dealers enter the investor-dealer market with at least one unit spare inventory capacity, any and all dealers are dealer-buyers in the market no matter how the dealer population is divided between the two types. Then, there will be the same $\theta_{DI} = n_B^D/n_L^{ON} = n^D/n_L^{ON}$ for all n^{LD} for which the Selling Equilibrium holds. Moreover, as every dealer buys and buys at the same probability for any n^{LD}, there must also be the same inventory holding among dealers, to be followed by the same n_S^D and therefore the same θ_{ID}. That is, as soon as the the Selling Equilibrium takes hold, the market's efficacy at intermediating trade among investors reaches its constrained best and cannot undergo any further improvement.

Inter-dealer Trading Prices For n^{LD} below the thresholds in Proposition 4a for which the Balanced Equilibrium holds, $p = \beta\left(V_1^{SD} - V_0^{SD}\right)$, where by (53) and (54) in the Appendix, respectively,

$$V_0^{SD} = W_0^{SD} + \mu(\theta_{DI})\,\frac{p - \beta U_L^{ON}}{2}, \tag{43}$$

$$V_1^{SD} = W_1^{SD} + \eta(\theta_{ID})\,\frac{\beta\left(U_H^{ON} - U^B\right) - p}{2}. \tag{44}$$

The first equation says that an S_0 has asset value equal to the value of his outside option W_0^{SD} plus the probability of trade times his share of the match surplus from trading with an investor-seller. The second equation, for the asset value of an S_1, can be interpreted similarly. By Proposition 4b, increases in n^{LD}, while the Balanced Equilibrium holds, cause θ_{DI} to go up and θ_{ID} to go down, from which dealers buy as well as sell at lower probabilities. To follow, both V_0^{SD} and V_1^{SD} tend to decline. The overall effect on p then appears ambiguous. In one set of quantitative analysis that we undertake, we find that p falls throughout, where V_1^{SD} declines more than V_0^{SD} does. This happens when the Balanced Equilibrium will turn into the Selling

Equilibrium and also when the Balanced Equilibrium will turn into the Buying Equilibrium.[25] We suspect that this overall negative effect is due to the tendency that as investors' matching rates go up amid the falling matching rates for dealers, high-valuation non-owners should stand to gain more from the faster acquisition of the asset than low-valuation owners from the faster disposition of the asset in the steady state.[26] Any larger increase in U^B than in U_L^{ON}, according to (43) and (44), should exert a negative effect on $V_1^{SD} - V_0^{SD}$ – an effect that apparently is of overriding importance in our quantitative analysis.

Proposition 4c *Holding fixed n^D, as n^{LD} rises, if and when the Balanced Equilibrium gives way to the Buying Equilibrium, p falls by a discrete amount down to zero; if and when the Balanced Equilibrium gives way to the Selling Equilibrium, p jumps up by a discrete amount and stays at the same level for all n^{LD}.*

The inter-dealer market price may vary with n^{LD} to the extent that either one or both of the market tightnesses vary in response to the change in n^{LD}. By Proposition 4b, as soon as the Selling Equilibrium takes hold, the two market tightnesses reach their respective maximum and minimum values. In the meantime, by Proposition 4c, p attains its highest possible value in equilibrium when it becomes anchored at $p = \beta\left(V_1^{LD} - V_0^{LD}\right)$ – the largest marginal benefit of inventory. A corollary of the Proposition and our previous quantitative analysis is that p can be non-monotonic with respect to increases n^{LD}, first decreasing while the Balanced Equilibrium holds, reaching the minimum at the transition to the Selling Equilibrium, and then going up by a discrete amount thereafter.[27]

Inter-dealer Trading Volume While the Balanced Equilibrium holds, the trading volume in the inter-dealer market is given by $\max\{m_0^{LD}, m_2^{LD}\}$. For given trading probabilities, L_0s and L_2s entering the inter-dealer market should be more numerous if large dealers simply constitute a bigger fraction of the dealer population. Furthermore, given n^{LD}, if dealers sell to and buy from investors both at lower probabilities, there should be fewer large dealers remaining as L_1s and more becoming either L_0s or L_2s at the closing of the investor-dealer market. In all, in the Balanced Equilibrium, TV should be increasing in n^{LD}.

While the Buying Equilibrium holds, the trading volume equals $m_0^{SD} + m_0^{LD}$. As small dealers are replaced one-for-one by large dealers, dealers leaving the investor-dealer market

[25] The numerical analyses assume $\eta(\theta) = 1 - e^{-\theta}$, $n^D = 1$, $e = 0.8$, $\delta = 0.1$, $\beta = 0.95$, and $A = 11$ for which the Balanced Equilibirum will turn into the Selling Equilibrium and $A = 12$ for which the Balanced Equilibrium will turn into the Buying Equilibrium. The equations for p, θ_{ID}, and θ_{DI} in the Balanced Equilibrium are given by (68), (88), and (89), respectively, in the Appendix.

[26] An investor-buyer benefits not just from there being a higher buying rate, but also from there being a higher selling rate since he can look forward to disposing of the unit he will hold later on faster if and when he suffers the liquidity shock. On the other hand, there is not any channel from which an investor-seller may benefit from a higher buying rate. In the steady state, then, U^B tends to increase more than U_L^{ON} does except perhaps when future payoffs are heavily discounted and when the investor-seller matching rate rises significantly more than the investor-buyer matching rate does.

[27] While the Balanced Equilibrium holds, $V_1^{LD} - V_0^{LD}$, like $V_1^{SD} - V_0^{SD}$, can also be decreasing in n^{LD}. If p were anchored at $\beta\left(V_1^{LD} - V_0^{LD}\right)$ for all n^{LD}, it should not increase at the transition from the Balanced to the Selling Equilibria at which point the two market tightness reach their respective maximum and minimum. But p is not anchored at $\beta\left(V_1^{LD} - V_0^{LD}\right)$ but at the lower $\beta\left(V_1^{SD} - V_0^{SD}\right)$ while the Balanced Equilibrium holds. The possible non-monotonicity arises from the change in the anchor of p at the transition.

with an empty inventory should fall in numbers since the large (but not the small) dealers may replenish any inventories they sell to investors in the same period of time by buying from other investors. Similarly, while the Selling Equilibrium holds, the trading volume, $m_1^{SD} + m_2^{LD}$, should fall when small dealers are replaced one-for-one by large dealers, as large dealers may restore the inventory capacities they forego during which they buy from investors by selling to other investors in the meantime.

Proposition 4d *Holding n^D fixed, the trading volume in the inter-dealer market is increasing in n^{LD} while the Balanced Equilibrium holds. Once the Buying or the Selling Equilibrium takes hold, the trading volume becomes decreasing in n^{LD}. TV is continuous at where the Balanced Equilibrium turns into either the Buying or the Selling Equilibrium and reaches the highest level equal to $e\left(1 - \frac{e}{n^D}\right)$ at the point of transition.*

Proposition 4d shows that for any level of asset supply, the inter-dealer market is least active when there is little diversity in the dealer population with n^{LD} either at the lowest or at the highest level. With more diversity as when n^{LD} is at some intermediate level, the market becomes more active. Trading in the inter-dealer market in our model then is mainly driven by the heterogeneity of dealers, rather than by dealers possessing more than a unit of inventory capacity.

Dealers' Bid and Ask Prices For brevity, in Propositions 3b and 4c, we have not extended the analysis to also checking how prices in the investor-dealer market may vary with A and n^{LD}. In Propositions A1 and A2 and the ensuing discussions in the Appendix, we show that the dealers' ask and bid prices do turn out to vary with A and n^{LD} in the just the same ways that the inter-dealer market price does.

4 Efficient Decentralized Market Trades

In this section, we study the problem of a social planner maximizing the discounted flow payoffs of investors over time given by,

$$W = \max \left\{ \sum_{t=0}^{\infty} \beta^t n_H^{ON}(t)\, \upsilon \right\}, \tag{45}$$

from the ownership of the asset, subject to the same search and matching frictions that agents in the model face.

A priori, the equilibrium trades in the frictional investor-dealer market are constrained efficient where any trades with a positive surplus, but only such trades, will take place with the terms of trade in the bilateral meetings reached via Nash Bargaining. Specifically, any investor-buyer and dealer-seller trade is efficient with the former, but not the latter, deriving the flow payoff υ in holding a unit of the asset. But then a dealer-seller becomes a dealer-seller in the first place only by acquiring the asset from an investor-seller. Then, any and all trades between an investor-seller and a dealer-buyer are also efficient.

This means that it suffices for us to ask how the planner may wish to allocate units of inventory among the dealers in each period after the investor-dealer trades are completed and whether the allocation coincides with the allocation that falls out from the inter-dealer market in equilibrium. The former allocation obviously should serve to enable high-valuation investors to acquire the asset most rapidly. Besides, it should also serve to enable units of the asset to be transferred from low-valuation investors to dealers the quickest, thereby facilitating the eventual sales to the high-valuation investors.

Lemma 3 *In the steady state of the planner's solution, units of inventory not held by investors are allocated to dealers to maximize the measures of dealer-sellers (dealers who hold inventory) and dealer-buyers (dealers who possess spare capacity). To maximize the measure of dealer-sellers, first allocate one unit each to either small or large dealers, and then allocate any remaining inventory to the large dealers. To maximize the measure of dealer-buyers, first allocate one unit each to large dealers, and then allocate any remaining inventory to either large or small dealers. The two objectives are then attained simultaneously by allocating inventory in the following order: (1) one unit each to large dealers; (2) if there remains any inventory, then one unit each to small dealers; (3) if there remains any inventory, one more unit each to large dealers.*

In the competitive inter-dealer market, inventories and spare capacities are allocated to dealers who value them the most – the very dealers who have the best use of them for trading with investors. Not surprisingly, the allocations coincide with the allocations for constrained efficiency in Lemma 3.

Proposition 5 *The decentralized market trades are constrained efficient.*

5 Conclusion and Discussion

In this paper, by means of a tractable random search model, we study inter-dealer trades among heterogeneous dealers in OTC markets motivated by inventory risk concerns. We depart from earlier such models by all assuming that all traders are risk neutral. Even so, the dealers benefit from trading among one another to eliminate the risks of carrying an insufficient inventory and an insufficient spare inventory capacity for their trading needs with investors. We show that the unique steady-state equilibrium is constrained efficient. The inter-dealer trading network that emerges endogenously resembles a core-periphery structure, with large dealers in the center, trading among themselves and with small dealers, who are in the periphery, trading with large core dealers only. The large core dealers each hold weakly more units of inventory in equilibrium than a small peripheral dealer does. These features match a number of the stylized facts documented in the literature.

The model yields a rich set of testable implications for future research. First and foremost, our analysis shows that the apparently obvious notion that large core dealers should provide inventory to small peripheral dealers insofar as the latter have larger inventory capacities only holds up when inventory is relatively abundant. But in this case, small dealers should need the inventory the least. In contrast, in our model, it is the small peripheral dealers who serve to provide inventory to the large core dealers when the latter need them the most. Besides, we show that the trading volume in the inter-dealer market does not rise monotonically with either the asset supply or the proportion of large dealers. Instead, trading in the inter-dealer market is most active when the asset supply is at a moderately low or at a moderately high level and when the dealer population is sufficiently diverse.

The model we studied in this paper is clearly a very special model, with numerous important simplifying assumptions. In closing, we comment on how our results might survive three generalizations that appear most warranted.

Inventory Capacity In our model, small dealers, in having a unit of inventory capacity, never gain from trading among themselves. The question then is if and how the core-periphery trading relationship in our setup survives the generalization where small dealers possess more than a unit of inventory capacity and thereby may gain by trading with one another. Consider, in particular, that the small dealers each possess a two-unit inventory capacity, while the large dealers each possess a three-unit inventory capacity. First, a ranking of the marginal value of inventory similar to that in Proposition 1 should remain – the large dealer should value the same unit of the asset more than a small dealer does, as the former has a greater spare capacity than the latter to satisfy future buying needs. On the other hand, the former should value an additional unit of the asset less than the latter if the two happen to possess the same spare capacity, as the former has a larger inventory than the latter to begin with at the same level of spare capacity. In Appendix 6.2, we discuss in detail the directions of trade and the formation of a core-periphery trading network in this setting. We find that while there can be trades between two small dealers, on balance, it remains true that when the asset supply is at a relatively low level, small dealers tend to provide inventory for large dealers, as small dealer-sellers turn out to outnumber small dealer-buyers in the inter-dealer market in such an environment. The converse holds when the asset supply is at a relatively high level. The trading direction is, then, largely persistent. The core-periphery trading relationship survives the generalization too, as a given large dealer trades with a greater diversity of other dealers and thus has more links than a given small dealer has.

The above suggests that the main results of our paper should also survive the generalization that there are more than two inventory capacities, as similar mechanisms should be operative to give rise to smaller-capacity dealers providing inventory (liquidity) for larger-capacity dealers when inventory (liquidity) is in greater demand. The inter-dealer trading network should retain a core-periphery structure as well since higher-capacity dealers should have more profitable trading opportunities with dealers of the same capacity than lower-capacity dealers have.

Matching Opportunity If each large dealer can hold up to two units in inventory and may possess up to two units of spare inventory capacity, perhaps a more natural assumption is that they can meet up to two investor-buyers and two investor-sellers in each period. The question then is how the ranking of the marginal benefits of inventory in Proposition 1 may be affected.

Now, if a large dealer has up to two matching opportunities with investor-buyers and with investor-sellers, respectively, the probabilities that the dealer meets at least one investor-buyer and one investor-seller should be weakly greater than the respective probabilities that a small dealer meets one investor-buyer and one investor-seller. We explain in Appendix 6.3, if in addition the matching technology exhibits diminishing returns in the sense that the probabilities that a large dealer is matched with as many as two investor-buyers and two investor-sellers are weakly lower than the respective probabilities that a small dealer is matched with one investor-buyer and one investor-seller, then the ranking in Proposition 1 is left intact.[28]

Competitive Inter-Dealer Market In reality, the inter-dealer market is better described as a decentralized market as documented in Li and Schürhoff (2014) and Henderschott, Li, Livdan and Schürhoff (2015), where it takes time and effort for a dealer to find a counterparty to trade with, in which case the dealer, by all means, has incentives to manage his inventory for future trading needs. By assuming dealers only have periodic, instead of continuous, access to the competitive inter-dealer market, the dealers in our model likewise have incentives to manage inventory. Where the incentives are similar, many features of the equilibrium in the present model, such as the core-periphery inter-dealer trading structure, should survive in an arguably richer model of a frictional inter-dealer market.

In Appendix 6.4, we present a model of a frictional inter-dealer market but one which is otherwise identical to the model studied in this paper. In particular, we replace the competitive inter-dealer in the model with a decentralized market in which the bilateral meetings between dealers take place, and that an exchange between two dealers is at a price reached by Nash Bargaining. The main takeaway from the analysis is that the counterpart to Proposition 1 on the ranking of the marginal benefits of inventory remains valid, meaning that exchanges between an L_0 and an L_2, between an L_0 and an S_1 and between an L_2 and an $S_{0.}$exhaust all profitable exchanges among dealers, as in the present model, from which a core-periphery trading structure emerges. In all, the core-periphery trading structure in our model does not hinge on a Walrasian inter-dealer market but is a generic feature of models assuming dealers having imperfect access to inter-dealer trades, however modeled, and that their inventory capacities differ.

The revised model, with a decentralized inter-dealer market, not surprisingly, has richer implications on prices. In particular, by the ranking of the marginal benefits of inventory, L_0s pay the highest price to investors, followed by S_0s, and then by L_1s. On the other hand, L_1s receive the highest price from investors, followed by S_1s, and then by L_2s. In the inter-dealer market, $p_{L_0,S_1} > p_{L_0,L_2} > p_{S_0,L_2}$, whereby large dealers unambiguously pay higher prices than small dealers.

We did not choose to pursue the analysis of the revised model in the main text as no further analytical results beyond the counterpart to Proposition 1 seem possible. Assuming a Walrasian inter-dealer market simplifies considerably and enables us to derive a rich set of analytical results.

[28] If the two matching outcomes for the large dealer are independent events, the probability that a large dealer meets at least one investor-buyer is $1-(1-\eta(\theta_{ID}))^2 = \eta(\theta_{ID})(2-\eta(\theta_{ID})) > \eta(\theta_{ID})$, the probability that the small dealer meets one investor-buyer. Similarly, the probability that a large dealer meets as many as two investor-buyers is $\eta(\theta_{ID})^2 < \eta(\theta_{ID})$.

6 Appendix

6.1 Dealers' Bid and Ask Prices

Lemma A1 *In all three types of equilibrium, the dealers' bid price; i.e., the price at which investors sell to dealers is given by*

$$p_{I_S} = \frac{1-\beta+\beta\eta\left(\theta_{DI}\right)}{2\left(1-\beta\right)+\beta\eta\left(\theta_{DI}\right)}p, \tag{46}$$

whereas the dealers' ask prices; i.e., the prices at which investors buy from dealers in the Selling, Balanced, and Buying Equilibria are given by, respectively,

$$p_{I_B} = \left(1+\frac{1-\beta}{\beta\eta\left(\theta_{ID}\right)}\right)p, \tag{47}$$

$$p_{I_B} = \left(1+\frac{\left(1-\beta+\beta\frac{\mu(\theta_{DI})}{2}\right)\left(4(1-\beta)+2\beta\eta(\theta_{DI})\right)-\beta^2\mu(\theta_{DI})\eta(\theta_{DI})}{\beta\eta(\theta_{ID})\left(4(1-\beta)+2\beta\eta(\theta_{DI})\right)}\right)p, \tag{48}$$

$$p_{I_B} = \frac{\beta\left(1-\beta\right)v}{\left(1-\beta+\beta\delta\right)\left(2\left(1-\beta\right)+\mu\left(\theta_{ID}\right)\beta\right)}. \tag{49}$$

Proposition A1 *For $e \leq n^{LD}$, as A increases from B_L+e/δ at which the Selling Equilibrium first holds, p_{I_S} and p_{I_B} are continuously decreasing in A. Once A reaches B_M+e/δ at which the Balanced Equilibrium begins to hold, there will be discrete falls in the two prices. While the Balanced Equilibrium holds, p_{I_S} is continuously decreasing in A. And then finally, when A rises up to and above B_L+e/δ, at which the Buying Equilibrium holds, there will be further discrete falls in the two prices – p_{I_S} all the way to zero and p_{I_B} to some positive value. Thereafter, the two prices do not vary with A any longer. For $e \in \left(n^{LD}, n^{LD}+\frac{n^{SD}}{2}\right]$ and that $A > B_M+e/\delta$ at which the Balanced Equilibrium begins to hold, p_{I_S} is likewise continuously decreasing in A.*

The Proposition leaves out how p_{I_B} may vary with A in the Balanced Equilibrium as it does not seem possible to sign $\partial p_{I_B}/\partial A$ in said equilibrium. Our numerical analyses do reveal, however, that p_{I_B} does decline in A while the Balanced Equilibrium holds,[29] just as p and p_{I_S} do.

For the smallest admissible n^{LD}, the market starts off in a Balanced Equilibrium, in which p, as our numerical analyses in the main text indicate, tends to decline with increases in n^{LD}. In the same numerical analyzes, we find that p_{I_S} and p_{I_B} follow the same tendency.

Proposition A2 *Holding fixed n^D, as n^{LD} rises, if and when the Balanced Equilibrium gives way to the Buying Equilibrium, both p_{I_S} and p_{I_B} fall by some discrete amount – p_{I_S} to zero and p_{I_B} to some positive value; if and when the Balanced Equilibrium gives way to the Selling Equilibrium, both p_{I_S} and p_{I_B} jump up by some discrete amount. In either the Buying or the Selling Equilibrium, the two prices do not vary with n^{LD}.*

[29] Under the same parameter configurations as for the numerical analyses preceding Proposition 4c, except that n^{LD} is fixed at 0.846.

6.2 Larger Inventory Capacity for Small Dealers

In this section, we discuss how the major results of the model survive the extension to allow small dealers to possess more than a unit of inventory capacity.

Suppose small dealers can hold up to two units of inventory and large dealers can hold up to three units. This larger inventory capacity is relevant only if a dealer may buy and sell up to two units of the asset in a period. The simplest extension is to assume that there are two types of investors – small and large, where the former, comprising a fraction α of the investor population, may each hold either zero or one unit, whereas the latter, comprising a fraction $1-\alpha$ of the investor population, may each hold either zero or two units. An exchange between a large investor-buyer (-seller) and a dealer-seller (-buyer) will only take place if the dealer happens to possess two units of the asset in inventory (spare inventory capacity).

The marginal value of inventory can be ranked in the following way in any steady-state equilibrium,

$$V_1^{LD} - V_0^{LD} \geq V_1^{SD} - V_0^{SD} \geq V_2^{LD} - V_1^{LD} \geq V_2^{SD} - V_1^{SD} \geq V_3^{LD} - V_2^{LD}.$$

In equilibrium, p must be equal to β times one of the above marginal values of inventory.

1a. $p = \beta\left(V_1^{LD} - V_0^{LD}\right)$ The buyers in the inter-dealer market are a fraction of L_0s and the sellers are S_1s, S_2s, L_2s, and L_3s.

1b. $p = \beta\left(V_1^{SD} - V_0^{SD}\right)$ **with a fraction of S_1s selling in the inter-dealer market** The buyers in the inter-dealer market are L_0s and the sellers are a fraction of S_1s, and all of S_2s, L_2s and L_3s.

In both (1a) and (1b), given that all small dealers who trade in the inter-dealer market (S_1s and S_2s) sell and they sell to L_0s, small dealers act to provide inventory to large dealers and the trading direction between small and large dealers is persistent. Also, since small dealers only sell in the inter-dealer market to large dealers but never trade among themselves, the trading network has the core-periphery structure.

2. $p = \beta\left(V_1^{SD} - V_0^{SD}\right)$ **with a fraction of S_0s buying in the inter-dealer market** The buyers in the inter-dealer market are L_0s and a fraction of S_0s, and the sellers are L_2s, S_2s and L_3s. That p is at a relatively high level should be due to a small asset supply, in which case it should be easier for dealers to sell to than to buy from investors. Then, there should be more S_0s than S_2s entering the inter-dealer market. But notice here that only a fraction of S_0s are buying. When this type of equilibrium first starts to hold, this fraction is arbitrarily close to zero. So there can still be fewer S_0 buyers than S_2 sellers. When this equilibrium turns into the equilibrium with $p = \beta\left(V_2^{LD} - V_1^{LD}\right)$ so that all S_0s are buying, by the argument below in the next case, there should be as many S_0s as S_2s. In between, we conjecture that there remains fewer S_0 buyers than S_2 sellers. Then, on balance, small dealers are providing inventory to large dealers and the trading direction is largely persistent.

In the inter-dealer market, a buyer (L_0 or S_0) can buy from an S_2, L_2 or L_3, and thus has three links; a seller (S_2, L_2 or L_3) can sell to an L_0 or some of the S_0s, and thus has two links.

As we argue above, S_2 sellers, each of whom has two links, should be more numerous than S_0 buyers (not all of S_0s, but those who actually buy in equilibrium), each of whom has three links. On the other hand, since it is easier to sell than to buy in the investor-dealer market, there should be fewer L_2 and L_3 sellers altogether, each of whom has two links, than L_0 buyers, each of whom has three links. Then, on average, active large dealers have more links in the inter-dealer market and the trading relationship is core-periphery.

3. $p = \beta\left(V_2^{LD} - V_1^{LD}\right)$ The buyers in the inter-dealer market are L_0s, S_0s, and possibly a fraction of L_1s. The sellers are S_2s, L_3s, and possibly a fraction of L_2s. Given that all small dealers leave the inter-dealer market with one unit of inventory whereas large dealers do so with either one or two units of inventory, when the investor-dealer market opens, all dealers are dealer-sellers as well as dealer-buyers. The latter implies that dealers meet investor-buyers and investor-sellers at the same probability. To see this, notice that in each period in the steady state, there have to be αe measure of small investors buying as well as selling, from which it follows that

$$\alpha e = n^D \eta(\theta_{ID})\alpha,$$

$$\alpha e = n^D \mu(\theta_{DI})\alpha.$$

The two conditions combine to yield $\eta(\theta_{ID}) = \mu(\theta_{DI})$, which further implies that $m_0^{SD} = m_2^{SD}$, given that

$$m_0^{SD} = \eta(\theta_{ID})\alpha\left(1 - \mu(\theta_{DI})\alpha\right) n_1^{SD},$$

$$m_2^{SD} = \mu(\theta_{DI})\alpha\left(1 - \eta(\theta_{ID})\alpha\right) n_1^{SD}.$$

In this case, overall small dealers provide neither inventory nor liquidity to large dealers when the asset supply is at an intermediate level.

Now, that $\eta(\theta_{ID}) = \mu(\theta_{DI})$ also implies $n_1^{LD} = n_2^{LD} = n^{LD}/2$ – in the steady state in each period, there have to be $(1-\alpha)\,e$ measure of large investors buying from large dealers holding a two-unit inventory as well as $(1-\alpha)\,e$ measure of large investors selling to large dealers holding a one-unit inventory, from which it follows that

$$(1-\alpha)\,e = n_2^{LD}\eta(\theta_{ID})\,(1-\alpha)\,,$$

$$(1-\alpha)\,e = n_1^{LD}\mu(\theta_{DI})\,(1-\alpha)\,.$$

Given $\eta(\theta_{ID}) = \mu(\theta_{DI})$, the two conditions hold at the same time only for $n_1^{LD} = n_2^{LD} = n^{LD}/2$. To follow then is $m_0^{LD} = m_3^{LD}$ given that

$$m_0^{LD} = n_1^{LD}\eta(\theta_{ID})\alpha\left(1 - \mu(\theta_{DI})\right) + n_2^{LD}\eta(\theta_{ID})\,(1-\alpha)\left(1 - \alpha\mu(\theta_{DI})\right),$$

$$m_3^{LD} = n_1^{LD}\mu(\theta_{DI})\,(1-\alpha)\left(1 - \alpha\eta(\theta_{ID})\right) + n_2^{LD}\mu(\theta_{DI})\alpha\left(1 - \eta(\theta_{ID})\right).$$

Then, in this equilibrium, with $m_0^{SD} = m_2^{SD}$ and $m_0^{LD} = m_3^{LD}$, L_1s and L_2s neither buy nor sell in the inter-dealer market.

Centrality of dealers Any dealer-buyer (S_0 or L_0) has two links (S_2 and L_3). Any dealer-seller (S_2 or L_3) has two links (S_0 and L_0). There is not any core-periphery structure, just as in our basic model when small dealers do not trade where $\eta(\theta_{ID}) = \mu(\theta_{DI})$, there is no clear-cut core-periphery structure to speak of.[30]

4. $p = \beta\left(V_2^{SD} - V_1^{SD}\right)$ **with a fraction of S_2s selling in the inter-dealer market** The situation is parallel to case 2. The net effect is that small dealers provide liquidity to large dealers.

5. $p = \beta\left(V_2^{SD} - V_1^{SD}\right)$ **with a fraction of S_1s buying in the inter-dealer market or** $p = \beta\left(V_3^{LD} - V_2^{LD}\right)$ The situation is parallel to case 1. Small dealers provide liquidity to large dealers.

6.3 Large dealers' meeting two investor-buyers and two investor-sellers

Suppose a large dealer may meet more than one investor-buyer and one investor-seller in one period with positive probability. We make the following assumptions regarding the meeting probabilities:

1. The probability that a large dealer meets at least one investor-buyer is weakly higher than the probability that a small dealer meets one investor-buyer.
2. The probability that a large dealer meets at least one investor-seller is weakly higher than the probability that a small dealer meets one investor-seller.
3. The probability that a large dealer meets two investor-buyers is weakly lower than the probability that a small dealer meets one investor-buyer.
4. The probability that a large dealer meets two investor-sellers is weakly lower than the probability that a small dealer meets one investor-seller.

First, consider the cost and benefit of acquiring the first unit of inventory in the inter-dealer market. The cost is higher for a small dealer. This occupied first unit of capacity matters to a small dealer if he meets one investor-seller and matters to a large dealer only if he meets two investor-sellers (if a large dealer meets only one investor-seller, he still has capacity to buy). By (4), the cost is higher for a small dealer. The benefit is higher for a large dealer – due to (1), the large dealer can sell the asset with weakly higher probability. Then, $V_1^{LD} - V_0^{LD} \geq V_1^{SD} - V_0^{SD}$. The inequality can be strict if either one of the relation in (1) or (4) is strict.

Second, consider the cost and benefit of filling in the last unit of spare capacity. The cost is higher for a large dealer. This occupied capacity matters to a small dealer if he meets one

[30] We may also measure the centrality by the probability of trade. An S_1 trades in the inter-dealer market as long as he does not end up also as an S_1 at the closing of the investor-dealer market. The probability is

$$1 - \mu(\theta_{DI})\eta(\theta_{ID})\alpha^2 - (1 - \mu(\theta_{DI})\alpha)\,(1 - \eta(\theta_{ID})\alpha)$$

An L_1 trade in the inter-dealer market if the dealer becomes either an L_0 or an L_3. The probability is

$$\eta(\theta_{ID})\alpha\,(1 - \mu(\theta_{DI})) + \mu(\theta_{DI})\,(1 - \alpha)\,(1 - \alpha\eta(\theta_{ID}))$$

There is the same probability of trade for an L_2. The two probabilities above cannot be ranked in general.

investor-seller and matters to a large dealer if he meets at least one investor-seller. By (2), the cost is higher for a large dealer. The benefit is higher for a small dealer. A small dealer can benefit from this additional unit of inventory if he meets one investor-buyer while a large dealer can benefit only if he meets two investor-buyers. Then, according to (3), a small dealer benefits more. Then, $V_1^{SD} - V_0^{SD} \geq V_2^{LD} - V_1^{LD}$. The inequality can be strict if either one of the relation in (2) or (3) is strict.

6.4 Frictional inter-dealer market

6.4.1 Search and Matching

It is more convenient to assume continuous time to analyze a model in which both the investor-dealer and the inter-dealer markets are decentralized. Assume that each agent discounts the future at the same rate r. The search and matching in the investor-dealer market, as in the model in the main text, takes place in two market segments, with respective market tightness θ_{ID} and θ_{DI}. Because all dealers potentially take part in inter-dealer trades, it is without loss of generality to assume that a given dealer meets another randomly selected dealer at a fixed rate α per unit of time. All notations have the same meanings as in the main model.

6.4.2 Value functions

Small dealers An S_0 can only buy. He meets an investor-seller at the rate $\mu(\theta_{DI})$ and another dealer at the rate α. Among all dealers that the S_0 may meet, there can be a potentially profitable exchange only if the counterparty is an L_1 or an L_2, since an exchange between an S_0 and an S_1 merely results in the two dealers switching states and there can be no trade between an S_0 and L_0. Assume all investor-dealer trades yield non-negative surpluses, a result to be verified later on. Then,

$$\begin{aligned} rV_0^{SD} &= \mu(\theta_{DI})\left(V_1^{SD} - V_0^{SD} - p_{S_0,I}\right) + \alpha \left\{ \frac{n_1^{LD}}{n^D} \max\left\{-p_{S_0,L_1} + V_1^{SD} - V_0^{SD}, 0\right\} \right. \\ &\quad \left. + \frac{n_2^{LD}}{n^D} \max\left\{-p_{S_0,L_2} + V_1^{SD} - V_0^{SD}, 0\right\} \right\}. \end{aligned}$$

An S_1 can only sell. He meets an investor-buyer at the rate $\eta(\theta_{ID})$. The S_1 may also sell to an L_0 or an L_1. There are no potentially profitable exchanges with any other dealers. Then,

$$\begin{aligned} rV_1^{SD} &= \eta(\theta_{ID})\left(p_{I,S_1} + V_0^{SD} - V_1^{SD}\right) + \alpha \left\{ \frac{n_0^{LD}}{n^D} \max\left\{p_{L_0,S_1} + V_0^{SD} - V_1^{SD}, 0\right\} \right. \\ &\quad \left. + \frac{n_1^{LD}}{n^D} \max\left\{p_{L_1,S_1} + V_1^{SD} - V_0^{SD}, 0\right\} \right\}. \end{aligned}$$

Large dealers An L_0 may buy from an investor-seller, an S_1, or an L_2. An exchange between the L_0 and an L_1 cannot be profitable since any such exchange just results in the two dealers switching states. Then,

$$
\begin{aligned}
rV_0^{LD} &= \mu\left(\theta_{DI}\right)\left(V_1^{LD}-V_0^{LD}-p_{L_0,I}\right)+\alpha\left\{\frac{n_1^{SD}}{n^D}\max\left\{-p_{L_0,S_1}+V_1^{LD}-V_0^{LD},0\right\}\right.\\
&\left.+\frac{n_2^{LD}}{n^D}\max\left\{-p_{L_0,L_2}+V_1^{LD}-V_0^{LD},0\right\}\right\}.
\end{aligned}
$$

An L_1 may buy from an investor-seller and sell to an investor-buyer. Among dealers, he may sell to an S_0, buy from an S_1, and either buy from or sell to another L_1. There can be no profitable exchange with an L_0 or L_2 since any such exchange just results in the two dealers switching states. Then,

$$
\begin{aligned}
rV_1^{LD} &= \mu\left(\theta_{DI}\right)\left(V_2^{LD}-V_1^{LD}-p_{L_1,I}\right)+\eta\left(\theta_{ID}\right)\left(p_{I,L_1}+V_0^{LD}-V_1^{LD}\right)\\
&+\alpha\left\{\frac{n_0^{SD}}{n^D}\max\left\{p_{S_0,L_1}+V_0^{LD}-V_1^{LD},0\right\}+\frac{n_1^{SD}}{n^D}\max\left\{-p_{L_1,S_1}+V_2^{LD}-V_1^{LD},0\right\}\right.\\
&\left.+\frac{n_1^{LD}}{n^D}\max\left\{-p_{L_1,L_1}+V_2^{LD}-V_1^{LD},p_{L_1,L_1}+V_0^{LD}-V_1^{LD},0\right\}\right\}.
\end{aligned}
$$

An L_2 may sell to an investor-buyer only. Among dealers, he may sell to an S_0 or an L_0. There can be no profitable exchange with an L_1 since any such exchange just results in the two dealers switching states. Then,

$$
\begin{aligned}
rV_2^{LD} &= \eta\left(\theta_{ID}\right)\left(p_{I,L_2}+V_1^{LD}-V_2^{LD}\right)+\alpha\left\{\frac{n_0^{SD}}{n^D}\max\left\{p_{S_0,L_2}+V_1^{LD}-V_2^{LD},0\right\}\right.\\
&\left.+\frac{n_0^{LD}}{n^D}\max\left\{p_{L_0,L_2}+V_1^{LD}-V_2^{LD},0\right\}\right\}.
\end{aligned}
$$

Investors An investor-buyer may buy from an S_1, an L_1, or an L_2. Then,

$$
rU^B=\mu\left(\theta_{ID}\right)\left(U_H^{ON}-U^B-\frac{n_1^{SD}}{n_S^D}p_{I,S_1}-\frac{n_1^{LD}}{n_S^D}p_{I,L_1}-\frac{n_2^{LD}}{n_S^D}p_{I,L_2}\right),
$$

where

$$
rU_H^{ON}=\upsilon+\delta\left(U_L^{ON}-U_H^{ON}\right).
$$

An investor-seller may sell to an S_0, an L_0, or an L_1. Then,

$$
rU_L^{ON}=\eta\left(\theta_{DI}\right)\left(\frac{n_1^{SD}}{n_B^D}p_{S_0,I}+\frac{n_0^{LD}}{n_B^D}p_{L_0,I}+\frac{n_1^{LD}}{n_B^D}p_{L_1,I}-U_L^{ON}\right).
$$

6.4.3 Prices and Surpluses

All terms of exchange are determined by Nash Bargaining where each agent in a given bilateral match possesses equal bargaining power.

An investor-buyer pays,

$$
p_{I,S_1}=\frac{U_H^{ON}-U^B+V_1^{SD}-V_0^{SD}}{2},
$$

$$p_{I,L_1} = \frac{U_H^{ON} - U^B + V_1^{LD} - V_0^{LD}}{2},$$

$$p_{I,L_2} = \frac{U_H^{ON} - U^B + V_2^{LD} - V_1^{LD}}{2},$$

respectively, to buy from an S_1, an L_1, and L_2. Such matches yield the respective surpluses,

$$z_{I,S_1} = U_H^{ON} - U^B - \left(V_1^{SD} - V_0^{SD}\right),$$

$$z_{I,L_1} = U_H^{ON} - U^B - \left(V_1^{LD} - V_0^{LD}\right),$$

$$z_{I,L_2} = U_H^{ON} - U^B - \left(V_2^{LD} - V_1^{LD}\right).$$

An investor-seller receives,

$$p_{S_0,I} = \frac{V_1^{SD} - V_0^{SD} + U_L^{ON}}{2},$$

$$p_{L_0,I} = \frac{V_1^{LD} - V_0^{LD} + U_L^{ON}}{2},$$

$$p_{L_1,I} = \frac{V_2^{LD} - V_1^{LD} + U_L^{ON}}{2},$$

respectively, from selling to an S_0, an L_0, and L_1. Such matches yield the respective surpluses,

$$z_{S_0,I} = V_1^{SD} - V_0^{SD} - U_L^{ON},$$

$$z_{L_0,I} = V_1^{LD} - V_0^{LD} - U_L^{ON},$$

$$z_{L_1,I} = V_2^{LD} - V_1^{LD} - U_L^{ON}.$$

An S_0 may buy from an L_1 and an L_2 at

$$p_{S_0,L_1} = \frac{V_1^{SD} - V_0^{SD} + V_1^{LD} - V_0^{LD}}{2},$$

$$p_{S_0,L_2} = \frac{V_1^{SD} - V_0^{SD} + V_2^{LD} - V_1^{LD}}{2},$$

respectively, Such matches yield the respective surpluses,

$$z_{S_0,L_1} = V_1^{SD} - V_0^{SD} - \left(V_1^{LD} - V_0^{LD}\right),$$

$$z_{S_0,L_2} = V_1^{SD} - V_0^{SD} - \left(V_2^{LD} - V_1^{LD}\right).$$

An S_1 may sell to an L_0 and an L_1 at

$$p_{L_0,S_1} = \frac{V_1^{LD} - V_0^{LD} + V_1^{SD} - V_0^{SD}}{2},$$

$$p_{L_1,S_1} = \frac{V_2^{LD} - V_1^{LD} + V_1^{SD} - V_0^{SD}}{2},$$

respectively. Such matches yield the respective surpluses,

$$z_{L_0,S_1} = V_1^{LD} - V_0^{LD} - \left(V_1^{SD} - V_0^{SD}\right),$$

$$z_{L_1,S_1} = V_2^{LD} - V_1^{LD} - \left(V_1^{SD} - V_0^{SD}\right).$$

An L_0 may buy from an L_2 at

$$p_{L_0,L_2} = \frac{V_1^{LD} - V_0^{LD} + V_2^{LD} - V_1^{LD}}{2},$$

whereas an exchange between two L_1s may take place at price

$$p_{L_1,L_1} = \frac{V_2^{LD} - V_1^{LD} + V_1^{LD} - V_0^{LD}}{2}.$$

Such matches yield the respective surpluses,

$$z_{L_0,L_2} = V_1^{LD} - V_0^{LD} - \left(V_2^{LD} - V_1^{LD}\right),$$

$$z_{L_1,L_1} = V_2^{LD} - V_1^{LD} - \left(V_1^{LD} - V_0^{LD}\right).$$

6.4.4 Analysis

Lemma A2 *Any investor-dealer match yields a non-negative surplus in any equilibrium in which both small and large dealers are active.*

If a given dealer-seller chooses not to sell to investor-buyers (IB), he must then sell to other dealers. If it is optimal for the dealer who is buying to sell to IB, it must be optimal for the given dealer to sell to IB as well – there cannot be any greater surplus of trade for the unit to pass to another dealer before the unit is sold to an IB. If it is not optimal for the dealer who is buying to sell to IB and if the given dealer does not find selling to any other agent optimal, the unit will never be passed on to an IB. In this case, there cannot be any surplus of trade at all for the given dealer-seller.

Proposition A3 $V_1^{LD} - V_0^{LD} \geq V_1^{SD} - V_0^{SD} \geq V_2^{LD} - V_1^{LD}$ *in any active equilibrium. The two equalities are strict unless* $z_{I,L_1} = 0$.

By Proposition A3 and if the inequalities are strict, only

$$z_{S_0,L_2} = V_1^{SD} - V_0^{SD} + V_1^{LD} - V_2^{LD} > 0,$$

$$z_{L_0,S_1} = V_1^{LD} - V_0^{LD} + V_0^{SD} - V_1^{SD} > 0,$$

$$z_{L_0,L_2} = V_1^{LD} - V_0^{LD} + V_1^{LD} - V_2^{LD} > 0,$$

whereas all other inter-dealer trades yield negative surpluses.

6.5 Proofs of Lemmas and Propositions

Proof of Lemma 1 Notice that

$$W_1^{SD} = W_0^{SD} + p, \tag{50}$$

$$W_0^{LD} = W_1^{LD} - p, \tag{51}$$

$$W_2^{LD} = W_1^{LD} + p. \tag{52}$$

The lemma then follows from (14)-(19).

Proof of Proposition 1 Substitute (21) and (23) and (50)-(52) into the value functions (6) and (7) and (11)-(13),

$$V_0^{SD} = W_0^{SD} + \frac{\mu(\theta_{DI})}{2}\left(p - \beta U_L^{ON}\right), \tag{53}$$

$$V_1^{SD} = W_0^{SD} + \left(1 - \frac{\eta(\theta_{ID})}{2}\right)p + \frac{\eta(\theta_{ID})}{2}\beta\left(U_H^{ON} - U^B\right), \tag{54}$$

$$V_0^{LD} = W_1^{LD} - \left(1 - \frac{\mu(\theta_{DI})}{2}\right)p - \frac{\mu(\theta_{DI})}{2}\beta U_L^{ON}, \tag{55}$$

$$V_1^{LD} = W_1^{LD} + \frac{\mu(\theta_{DI}) - \eta(\theta_{ID})}{2}p - \frac{\mu(\theta_{DI})}{2}\beta U_L^{ON} + \frac{\eta(\theta_{ID})}{2}\beta\left(U_H^{ON} - U^B\right), \tag{56}$$

$$V_2^{LD} = W_1^{LD} + \left(1 - \frac{\eta(\theta_{ID})}{2}\right)p + \frac{\eta(\theta_{ID})}{2}\beta\left(U_H^{ON} - U^B\right). \tag{57}$$

We can then calculate

$$\left(V_1^{LD} - V_0^{LD}\right) - \left(V_1^{SD} - V_0^{SD}\right) = \frac{\mu(\theta_{DI})}{2}\left(p - \beta U_L^{ON}\right), \tag{58}$$

$$\left(V_1^{SD} - V_0^{SD}\right) - \left(V_2^{LD} - V_1^{LD}\right) = \frac{\eta(\theta_{ID})}{2}\left(\beta\left(U_H^{ON} - U^B\right) - p\right). \tag{59}$$

Notice that the terms inside the brackets in (58) and (59) denote, respectively, the surpluses of trade between an investor-seller and any dealer-buyer and between an investor-buyer and any dealer-seller in (20) and (22). If either of the two is negative, there cannot be any trade in equilibrium between investors and dealers in the steady state.

Proof of Lemma 2 Substitute (21) into (3) and rearrange,

$$U_L^{ON} = \frac{\frac{\eta(\theta_{DI})}{2}}{1 - \beta + \beta\frac{\eta(\theta_{DI})}{2}}p. \tag{60}$$

Substitute the equation into (2) and rearrange,

$$U_H^{ON} = \frac{\left(1 - \beta + \frac{\eta(\theta_{DI})}{2}\beta\right)\upsilon + \beta\delta\frac{\eta(\theta_{DI})}{2}p}{(1 - \beta + \beta\delta)\left(1 - \beta + \frac{\eta(\theta_{DI})}{2}\beta\right)} \tag{61}$$

Substitute (23) into (1) and rearrange,

$$U^B = \frac{\frac{\mu(\theta_{ID})}{2}\left(\beta U_H^{ON} - p\right)}{1-\beta+\beta\frac{\mu(\theta_{ID})}{2}}. \tag{62}$$

Substituting from (61),

$$U^B = \frac{\mu\left(\theta_{ID}\right)}{2}\frac{\beta\left(1-\beta+\frac{\eta(\theta_{DI})}{2}\beta\right)\upsilon-(1-\beta)\left(1-\beta+\beta\delta+\frac{\eta(\theta_{DI})}{2}\beta\right)p}{(1-\beta+\beta\delta)\left(1-\beta+\beta\frac{\mu(\theta_{ID})}{2}\right)\left(1-\beta+\frac{\eta(\theta_{DI})}{2}\beta\right)}. \tag{63}$$

Then, by (61) and (63),

$$U_H^{ON} - U^B = \frac{\left(\frac{\mu(\theta_{ID})}{2}(1-\beta+\beta\delta)\left(1-\beta+\frac{\eta(\theta_{DI})}{2}\beta\right)+\beta\delta\frac{\eta(\theta_{DI})}{2}(1-\beta)\right)p+\left(1-\beta+\frac{\eta(\theta_{DI})}{2}\beta\right)(1-\beta)\upsilon}{(1-\beta+\beta\delta)\left(1-\beta+\frac{\mu(\theta_{ID})}{2}\beta\right)\left(1-\beta+\frac{\eta(\theta_{DI})}{2}\beta\right)} \tag{64}$$

Set $p = \beta\left(V_1^{LD} - V_0^{LD}\right)$ and by (55) and (56),

$$p = \frac{\beta\frac{\eta(\theta_{ID})}{2}}{1-\beta+\beta\frac{\eta(\theta_{ID})}{2}}\beta\left(U_H^{ON}-U^B\right). \tag{65}$$

Then use (64) to obtain

$$p = \frac{\beta^2\frac{\eta(\theta_{ID})}{2}\left(1-\beta+\frac{\eta(\theta_{DI})}{2}\beta\right)\upsilon}{\left((1-\beta+\beta\delta)\left(1-\beta+\beta\frac{\mu(\theta_{ID})}{2}\right)+\beta\frac{\eta(\theta_{ID})}{2}(1-\beta)\right)\left(1-\beta+\frac{\eta(\theta_{DI})}{2}\beta\right)+\delta\beta^2\frac{\eta(\theta_{ID})}{2}(1-\beta)}. \tag{66}$$

Given the positivity of p in (66) and by (60) and (65),

$$0 < \beta U_L^{ON} < p < \beta\left(U_H^{ON} - U^B\right).$$

Next, set $p = \beta\left(V_1^{SD} - V_0^{SD}\right)$ and by (53) and (54),

$$p = \frac{\frac{\eta(\theta_{ID})}{2}\beta^2\left(U_H^{ON}-U^B\right)+\frac{\mu(\theta_{DI})}{2}\beta^2 U_L^{ON}}{1-\beta+\beta\left(\frac{\eta(\theta_{ID})}{2}+\frac{\mu(\theta_{DI})}{2}\right)}. \tag{67}$$

Then use (60) and (64) to obtain

$$p = \frac{\frac{\eta(\theta_{ID})}{2}\beta^2\left(1-\beta+\frac{\eta(\theta_{DI})}{2}\beta\right)\upsilon}{\left(1-\beta+\frac{\eta(\theta_{DI})}{2}\beta+\beta\frac{\mu(\theta_{DI})}{2}\right)\left(1-\beta+\frac{\mu(\theta_{ID})}{2}\beta\right)(1-\beta+\beta\delta)+\left(1-\beta+\beta\delta+\frac{\eta(\theta_{DI})}{2}\beta\right)\beta\frac{\eta(\theta_{ID})}{2}(1-\beta)}. \tag{68}$$

Given the positivity of p in (68) and by (60) and (67),

$$0 < \beta U_L^{ON} < p < \beta\left(U_H^{ON} - U^B\right).$$

Finally, set $p = \beta\left(V_2^{LD} - V_1^{LD}\right)$ and by (56) and (57),

$$p = \frac{\beta\frac{\mu(\theta_{DI})}{2}}{1 - \beta + \beta\frac{\mu(\theta_{DI})}{2}}\beta U_L^{ON}. \tag{69}$$

With (60),

$$p = \beta U_L^{ON} = 0.$$

Next, by (64),

$$U_H^{ON} - U^B = \frac{(1-\beta)\upsilon}{(1-\beta+\beta\delta)\left(1-\beta+\frac{\mu(\theta_{ID})}{2}\beta\right)} > 0. \tag{70}$$

Thus,

$$0 = \beta U_L^{ON} = p < \beta\left(U_H^{ON} - U^B\right).$$

Proof of Proposition 2 Before proceeding to prove the Proposition, it is useful to establish the following.

Remark 1 For $x \leq 1$, $x < \eta^{-1}(x)$.
Proof. Given $\mu(x) = \eta(x)/x$ and that $\mu(x) < 1$, $\eta(x) < x$. And then for $x \leq 1$, the last condition implies $x < \eta^{-1}(x)$.

Remark 2 For $x \geq 1$, $x > \mu^{-1}\left(\frac{1}{x}\right)$.
Proof. Given that $\eta(x) = x\mu(x)$ and that $\eta(x) < 1$, $\mu(x) < \frac{1}{x}$. And then for $x \geq 1$, the last condition implies $x > \mu^{-1}\left(\frac{1}{x}\right)$.

Remark 3 For $e \leq n$, $\frac{n}{\mu^{-1}\left(\frac{e}{n}\right)}$ is decreasing in n.
Proof. By differentiation.

Now, to begin proving the Proposition, we start with manipulating (29)-(31) to obtain,

$$n_H^{ON} = \frac{e}{\delta}, \tag{71}$$

$$n_L^{ON} = \frac{e}{\eta(\theta_{DI})}, \tag{72}$$

$$n_B^{I} = \frac{e}{\mu(\theta_{ID})}. \tag{73}$$

Selling Equilibrium In the Selling Equilibrium, $n_2^{LD} = n_1^{SD} = 0$ and $n_0^{SD} = n^{SD}$. Then, together with (72) and (73), the two market tightness equations, (27) and (28), specialize to, respectively,

$$\eta(\theta_{ID}) = \frac{e}{n_1^{LD}}, \tag{74}$$

$$\mu(\theta_{DI}) = \frac{e}{n^D}. \tag{75}$$

By (26), (71), (72), and that $n_1^{SD} = n_2^{LD} = 0$ in the Selling Equilibrium,

$$n_1^{LD} = A - \frac{e}{\delta} - \frac{e}{\eta\left(\theta_{DI}\right)}. \tag{76}$$

Substitute the equation into (74) and rearrange,

$$\eta(\theta_{ID}) = \frac{\eta(\theta_{DI})e}{(A - \frac{e}{\delta})\eta(\theta_{DI}) - e}. \tag{77}$$

Once θ_{DI} is known from (75), the above uniquely gives θ_{ID}. For θ_{DI} from (75) to be a valid equilibrium, it has to be such that the resulting: (a) $\eta(\theta_{ID}) \in (0,1)$, as given by (77) and (b) $n_1^{LD} \in (e, n^{LD}]$, as given by (76). For (b) to be satisfied, $e < n^{LD}$ must hold.

By (77), for $\eta(\theta_{ID}) \in (0,1)$,

$$\frac{e}{\eta\left(\theta_{DI}\right)} < A - \frac{e}{\delta} - e. \tag{RS.1}$$

Substituting from (75) and rearranging, $(RS.1)$ holds if

$$A - \frac{e}{\delta} > e + \frac{n^D}{\mu^{-1}\left(\frac{e}{n^D}\right)} = B_S. \tag{AS.1}$$

Given that $\eta(\theta_{ID}) < 1$, $n_1^{LD} > e$ holds for sure. By (76), for $n_1^{LD} \leq n^{LD}$,

$$\frac{e}{\eta\left(\theta_{DI}\right)} \geq A - \frac{e}{\delta} - n^{LD}. \tag{RS.2}$$

Given θ_{DI} from (75), the above becomes,

$$A - \frac{e}{\delta} \leq n^{LD} + \frac{n^D}{\mu^{-1}\left(\frac{e}{n^D}\right)} = B_M. \tag{AS.2}$$

Notice that for $(RS.1)$ and $(RS.2)$ to be satisfied at the same time, it has to be such that $e < n^{LD}$, in which case (75) and the two conditions $(AS.1)$ and $(AS.2)$ are guaranteed to be well-defined.

Substituting (33), (34), and (36) into the inter-dealer market equilibrium condition (37),

$$\left(n^{SD} + n_1^{LD}\right)\mu(\theta_{DI}) \leq n_0^{LD}\left(1 - \mu(\theta_{DI})\right) + n_1^{LD}\eta(\theta_{ID}).$$

And then with (74)-(77), the condition can be shown to simplify to $(RS.2)$.

To sum up, the Selling Equilibrium holds if and only if $(AS.1)$ and $(AS.2)$ hold; i.e., $A - \frac{e}{\delta} \in (B_S, B_M]$, in addition to $e < n^{LD}$.

Buying Equilibrium In the Buying Equilibrium, $n_0^{LD} = n_0^{SD} = 0$ and $n_1^{SD} = n^{SD}$. Then, together with (72) and (73), the two market tightness equations, (27) and (28), specialize to, respectively,

$$\eta\left(\theta_{ID}\right) = \frac{e}{n^D}, \tag{79}$$

$$\mu\left(\theta_{DI}\right)=\frac{e}{n_1^{LD}}. \tag{80}$$

By (26), (71), (72), and that $n_0^{SD}=n_0^{LD}=0$ in the Buying Equilibrium,

$$n_1^{LD}=n^{SD}+2n^{LD}-A+\frac{e}{\delta}+\frac{e}{\eta(\theta_{DI})}. \tag{81}$$

Substitute the equation into (80),

$$e(\theta_{DI}-1)-\eta(\theta_{DI})\left(n^{SD}+2n^{LD}-A+\frac{e}{\delta}\right)=0, \tag{82}$$

which is an equation in θ_{DI} alone. It is straightforward to verify that there is a unique positive solution of θ_{DI} to the equation, and that the LHS is positive (negative) for θ_{DI} above (below) the solution of the equation. For the solution to be a valid equilibrium, it has to be such that the resulting $n_1^{LD}\in(e,n^{LD}]$, as given by (81). Then $e<n^{LD}$ must be satisfied.

Rearranging (81), $n_1^{LD}\leq n^{LD}$ if and only if

$$\frac{e}{\eta(\theta_{DI})}\leq A-\frac{e}{\delta}-n^D. \tag{RB.1}$$

A necessary condition for the equation to hold is that

$$A-\frac{e}{\delta}-n^D\geq e. \tag{AB1.a}$$

Then, $(RB.1)$ holds if the LHS of (82) is non-positive when evaluated at $\theta_{DI}=\eta^{-1}\left(\frac{e}{A-\frac{e}{\delta}-n^D}\right)$. Where $e<n^{LD}$, which is necessary for the RHS of (80) and also guarantees the RHS of (79) to be bounded below one, the condition reads

$$A-\frac{e}{\delta}\geq n^D+\frac{n^{LD}}{\mu^{-1}\left(\frac{e}{n^{LD}}\right)}=B_L, \tag{AB1.b}$$

which subsumes $(AB1.a)$.

Rearranging (81), $n_1^{LD}>e$ if and only if

$$A-\frac{e}{\delta}+e-n^{SD}-2n^{LD}<\frac{e}{\eta(\theta_{DI})} \tag{RB.2}$$

Hence, if

$$A-\frac{e}{\delta}-n^{SD}-2n^{LD}\leq 0,$$

then $(RB.2)$ holds for sure. Otherwise, the condition holds if the LHS of (82) is positive when evaluated at $\theta_{DI}=\eta^{-1}\left(\frac{e}{A-\frac{e}{\delta}+e-n^{SD}-2n^{LD}}\right)$; i.e.,

$$\eta^{-1}\left(\frac{e}{A-\frac{e}{\delta}+e-n^{SD}-2n^{LD}}\right)>\frac{e}{A-\frac{e}{\delta}+e-n^{SD}-2n^{LD}}.$$

But the condition is guaranteed to hold by Remark 1.

Substituting (32), (34), and (36) into the inter-dealer market equilibrium condition (41),

$$\left(n^{SD}+n_1^{LD}\right)\eta(\theta_{ID})\leq n_1^{LD}\mu(\theta_{DI})+n_2^{LD}\left(1-\eta(\theta_{ID})\right).$$

Then, by (79)-(82), the condition can be shown to simplify to $(RB.1)$.

To sum up, the Buying Equilibrium holds if and only if $(AB.1b)$ holds; i.e., $A-\frac{e}{\delta}\geq B_L$, in addition to $e<n^{LD}$.

Balanced Equilibrium In the Balanced Equilibrium, $n_0^{LD} = n_2^{LD} = 0$ and $n_1^{LD} = n^{LD}$. Then, together with (72) and (73), the two market tightness equations, (27) and (28), specialize to, respectively,

$$\eta(\theta_{ID}) = \frac{e}{n_1^{SD} + n^{LD}}, \tag{84}$$

$$\mu(\theta_{DI}) = \frac{e}{n_0^{SD} + n^{LD}}. \tag{85}$$

By (26), (71), and (72), and that $n_0^{LD} = n_2^{LD} = 0$ in the Balanced Equilibrium,

$$n_1^{SD} = A - n^{LD} - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})}, \tag{86}$$

and therefore

$$n_0^{SD} = n^D - A + \frac{e}{\delta} + \frac{e}{\eta(\theta_{DI})}. \tag{87}$$

Substituting (86) and (87) into (84) and (85), respectively, and rearranging,

$$\eta(\theta_{ID}) = \frac{\eta(\theta_{DI})\, e}{(A - e/\delta)\,\eta(\theta_{DI}) - e}, \tag{88}$$

$$e(\theta_{DI} - 1) - \eta(\theta_{DI})\left(n^D + n^{LD} - A + \frac{e}{\delta}\right) = 0, \tag{89}$$

which are respectively the same equations that give θ_{ID} in the Selling Equilibrium in (77) and θ_{DI} in the Buying Equilibrium in (82).

For now, we restrict attention to where $e < \frac{n^D + n^{LD}}{2}$. Later on, we will verify that the condition is necessary for the existence of the Balanced Equilibrium. Now, for the solution of (89) to be a valid equilibrium, it has to be such that (a) n_0^{SD} as given by (87) satisfies $n_0^{SD} \in \left[0, n^{SD}\right]$ for $e < n^{LD}$ and $n_0^{SD} \in \left(e - n^{LD}, n^D - e\right)$ for $e \in \left[n^{LD}, \frac{n^D + n^{LD}}{2}\right)$ and (b) $\eta(\theta_{ID}) \in (0,1)$, as given by (88).

By (87), where $e < n^{LD}$, for $n_0^{SD} \geq 0$,

$$\frac{e}{\eta(\theta_{DI})} \geq A - \frac{e}{\delta} - n^D \tag{RBA.1}$$

has to hold. The condition is guaranteed to hold if

$$A - \frac{e}{\delta} - n^D \leq e. \tag{ABA.1a}$$

Otherwise, $(RBA.1)$ can only hold if the LHS of (89) is non-negative when evaluated at $\theta_{DI} = \eta^{-1}\left(\frac{e}{A - \frac{e}{\delta} - n^D}\right)$; i.e.,

$$A - \frac{e}{\delta} \leq n^D + \frac{n^{LD}}{\mu^{-1}\left(\frac{e}{n^{LD}}\right)} = B_L. \tag{ABA.1b}$$

Note that $(RBA.1)$ holds if either $(ABA.1a)$ or $(ABA.1b)$ is satisfied. Given that $\frac{n^{LD}}{\mu^{-1}\left(\frac{e}{n^{LD}}\right)} < e$ by Remark 2, however, the latter condition subsumes the former one to begin with. Next, for $n_0^{SD} \leq n^{SD}$, by (87),

$$\frac{e}{\eta(\theta_{DI})} \leq A - \frac{e}{\delta} - n^{LD}. \tag{RBA.2}$$

The condition holds if the LHS of (89) is non-positive when evaluated at $\theta_{DI} = \eta^{-1}\left(\frac{e}{A-\frac{e}{\delta}-n^{LD}}\right)$; i.e.,

$$A - \frac{e}{\delta} \geq n^{LD} + \frac{n^D}{\mu^{-1}\left(\frac{e}{n^D}\right)} = B_M. \tag{ABA.2}$$

Where $e \in \left[n^{LD}, \frac{n^D+n^{LD}}{2}\right)$, for $n_0^{SD} > e - n^{LD}$,

$$\frac{e}{\eta\left(\theta_{DI}\right)} > A - \frac{e}{\delta} - n^D - n^{LD} + e \tag{RBA.3}$$

has to hold. The condition is guaranteed to hold if

$$A - \frac{e}{\delta} - n^D - n^{LD} \leq 0.$$

Otherwise, $(RBA.3)$ can only hold if the LHS of (89) is positive when evaluated at $\theta_{DI} = \eta^{-1}\left(\frac{e}{A-\frac{e}{\delta}-n^D-n^{LD}+e}\right)$; i.e.,

$$\eta^{-1}\left(\frac{e}{A-\frac{e}{\delta}-n^D-n^{LD}+e}\right) - \frac{e}{A-\frac{e}{\delta}-n^D-n^{LD}+e} > 0.$$

This inequality is met for sure by Remark 1, as $A-\frac{e}{\delta}-n^D-n^{LD} > 0$ implies $\frac{e}{A-\frac{e}{\delta}-n^D-n^{LD}+e} < 1$. Next, for $n_0^{SD} < n^D - e$, by (87),

$$\frac{e}{\eta\left(\theta_{DI}\right)} < A - \frac{e}{\delta} - e, \tag{RBA.4}$$

By (88), the condition for $\eta\left(\theta_{ID}\right) \in (0,1)$ is the same condition as $(RBA.4)$. The condition holds if the LHS of (89) is negative when evaluated at $\theta_{DI} = \eta^{-1}\left(\frac{e}{A-\frac{e}{\delta}-e}\right)$; i.e.,

$$\frac{n^D+n^{LD}-e}{A-\frac{e}{\delta}-e} - \eta^{-1}\left(\frac{e}{A-\frac{e}{\delta}-e}\right) > 0. \tag{90}$$

The condition can only hold for $e < \frac{n^D+n^{LD}}{2}$, justifying our previous claim that the Balanced Equilibrium can only hold for e bounded from below the given value, because, otherwise,

$$\frac{n^D+n^{LD}-e}{A-\frac{e}{\delta}-e} \leq \frac{e}{A-\frac{e}{\delta}-e} < \eta^{-1}\left(\frac{e}{A-e/\delta-e}\right),$$

where the last inequality is by Remark 1. Given $e < \frac{n^D+n^{LD}}{2}$, (90) is equivalent to

$$A - \frac{e}{\delta} > e + \frac{n^D+n^{LD}-e}{\mu^{-1}\left(\frac{e}{n^D+n^{LD}-e}\right)} = \mathcal{B}_M. \tag{ABA.4}$$

The condition for there to be more sellers than buyers among large dealers in the inter-dealer market (38), by (34) and (36), and $n_0^{LD} = n_2^{LD} = 0$ in the Balanced Equilibrium, simplifies to

$$\mu\left(\theta_{DI}\right) \geq \eta\left(\theta_{ID}\right).$$

By (88),

$$\mu\left(\theta_{DI}\right)-\eta\left(\theta_{ID}\right)=\mu\left(\theta_{DI}\right)\frac{\left(A-e/\delta\right)\eta\left(\theta_{DI}\right)-e-e\theta_{DI}}{\left(A-e/\delta\right)\eta\left(\theta_{DI}\right)-e}.$$

The denominator is guaranteed positive for $\eta\left(\theta_{ID}\right)\in[0,1]$. The expression then has the same sign as the numerator; i.e.,

$$\mu\left(\theta_{DI}\right)-\eta\left(\theta_{ID}\right)\geq 0\Leftrightarrow\eta\left(\theta_{DI}\right)\left(A-\frac{e}{\delta}\right)-e-e\theta_{DI}\geq 0.$$

Rewrite (89) as

$$\eta\left(\theta_{DI}\right)\left(A-\frac{e}{\delta}\right)-e-e\theta_{DI}=\eta\left(\theta_{DI}\right)\left(n^{D}+n^{LD}\right)-2e\theta_{DI}. \tag{92}$$

We seek conditions on how the two sides of the equation meet at a non-negative value.

Properties of $\eta\left(\theta\right)\left(A-\frac{e}{\delta}\right)-e-e\theta$

1. equal to $-e$ at $\theta=0$,
2. tends to negative infinity as $\theta\to\infty$,
3. given condition $(ABA.4)$, so that $A-\frac{e}{\delta}-e>0$, is inverted-U,
4. if $\max\left\{\eta\left(\theta\right)\left(A-\frac{e}{\delta}\right)-e-e\theta\right\}>0$, rises above zero for a range of θ.

Properties of $\eta\left(\theta\right)\left(n^{D}+n^{LD}\right)-2e\theta$

1. equal to 0 at $\theta=0$,
2. tends to negative infinity as $\theta\to\infty$.
3. For $e<\frac{n^{D}+n^{LD}}{2}$, is inverted-U.

Given these properties of the two sides of (92), the RHS is greater than the LHS before the two sides meet, whereas the LHS is less than the RHS thereafter. Then, if at where the RHS vanishes, i.e.,

$$\theta=\mu^{-1}\left(\frac{2e}{n^{D}+n^{LD}}\right),$$

the LHS is non-negative; i.e.,

$$A-\frac{e}{\delta}\geq\frac{n^{D}+n^{LD}}{2}+\frac{\frac{n^{D}+n^{LD}}{2}}{\mu^{-1}\left(\frac{2e}{n^{D}+n^{LD}}\right)}=S, \tag{ABA.S}$$

then the meeting point is where the two sides are non-negative.

If $(ABA.S)$ holds, the relevant inter-dealer market equilibrium condition is (39), which becomes

$$n^{LD}\left(\mu\left(\theta_{DI}\right)-\eta\left(\theta_{ID}\right)\right)\leq\left(n_{0}^{SD}\left(1-\mu\left(\theta_{DI}\right)\right)+n_{1}^{SD}\eta\left(\theta_{ID}\right)\right),$$

after substituting in (32), (34), and (36). By (86)-(88), the condition becomes

$$\left(n^{D}-A+\frac{e}{\delta}+\frac{e}{\eta\left(\theta_{DI}\right)}\right)\left(1-\mu\left(\theta_{DI}\right)\right)+\left(\frac{e}{\mu\left(\theta_{DI}\right)}-n^{LD}\right)\mu\left(\theta_{DI}\right)\geq 0. \tag{RBA.5}$$

Rewrite (89) as

$$\frac{e}{\mu(\theta_{DI})} - n^{LD} = n^D - A + \frac{e}{\delta} + \frac{e}{\eta(\theta_{DI})}$$

The LHS of ($RBA.5$) is a weighted average of the two terms in this equation. Thus, if the equation holds where the two sides are non-negative, ($RBA.5$) must hold. In turn, in case $e < n^{LD}$ and if ($RBA.1$) holds, under which $n_0^{SD} \geq 0$, and in case $e \in \left[n^{LD}, \frac{n^D+n^{LD}}{2}\right)$ and if ($RBA.3$) holds, under which $n_0^{SD} > e - n^{LD}$, the RHS of the equation is guaranteed non-negative.

If ($ABA.S$) holds in reverse, the relevant inter-dealer market equilibrium condition is (40), which becomes

$$n^{LD}(\eta(\theta_{ID}) - \mu(\theta_{DI})) \leq n_0^{SD}\mu(\theta_{DI}) + n_1^{SD}(1 - \eta(\theta_{ID})),$$

after substituting in (33), (34), and (36). By (86)-(88), the condition becomes

$$\left(A - n^{LD} - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})}\right)(1 - \mu(\theta_{DI})) + \left(n^D - \frac{e}{\mu(\theta_{DI})}\right)\mu(\theta_{DI}) \geq 0. \qquad \text{(RBA.6)}$$

Rewrite (89) as

$$A - n^{LD} - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})} = n^D - \frac{e}{\mu(\theta_{DI})}.$$

The LHS of ($RBA.6$) is a weighted average of the two terms in this equation. Thus if the equation holds at the point where the two sides are non-negative, ($RBA.6$) must hold. In turn, in case $e < n^{LD}$ and if ($RBA.2$) holds, under which $n_0^{SD} \leq n^{SD}$ and in case $e \in \left[n^{LD}, \frac{n^D+n^{LD}}{2}\right)$ and if ($RBA.4$) holds, under which $n_0^{SD} < n^D - e$, the LHS of the equation is guaranteed non-negative.

Notice that in case $e < n^{LD}$, ($RBA.2$) is a more stringent condition than ($RBA.4$). Then, for θ_{ID} and θ_{DI} defined by (88) and (89) to be a valid Balanced Equilibrium, it suffices that ($ABA.1b$) and ($ABA.2$) hold; i.e., $A - \frac{e}{\delta} \in [B_M, B_L]$. Otherwise for $e \in \left[n^{LD}, \frac{n^D+n^{LD}}{2}\right)$, the equilibrium holds under ($ABA.4$); i.e., $A - \frac{e}{\delta} > \mathcal{B}_M$. In either case, for $A - \frac{e}{\delta} \leq S$, small dealers sell in equilibrium; otherwise small dealers buy.

Ranking of the Bounds That $B_S \leq B_M$ follows from $e \leq n^{LD}$, whereas that $B_M \leq S \leq B_L$ follows from $n^{LD} \leq n^D$ and Remark 3. That $\mathcal{B}_M \leq S$ follows from $e \leq n^{LD} + \frac{n^{SD}}{2}$ and Remark 3.

Proof of Proposition 3a In the Selling Equilibrium, θ_{DI} is implicitly given by (75), in which A is absent. By (77), θ_{ID} is decreasing in A given that θ_{DI} does not vary with A.

In the Balanced Equilibrium, θ_{DI} is implicitly given by (89), the solution to which is at a point where the LHS of the equation is increasing. In the meantime, the LHS of the equation is increasing in A. Then, $\partial\theta_{DI}/\partial A < 0$. To evaluate the effect of A on θ_{ID}, first rewrite (89) as

$$A - \frac{e}{\delta} = n^D + n^{LD} - \frac{e(\theta_{DI} - 1)}{\eta(\theta_{DI})}.$$

Then substitute the equation into (88) to yield

$$\eta\left(\theta_{ID}\right)=\frac{\eta\left(\theta_{DI}\right)e}{\left(n^{D}+n^{LD}\right)\eta\left(\theta_{DI}\right)-e\theta_{DI}},$$

the RHS of which is increasing in θ_{DI} due to the concavity of η. Then, θ_{ID} must be decreasing in A.

In the Buying Equilibrium, θ_{ID} is implicitly given by (79), in which A is absent, whereas θ_{DI} is given by the same equation that defines θ_{DI} in the Balanced Equilibrium.

The continuity can be established by verifying that the equations for θ_{DI} and θ_{ID} for one equilibrium type coincide with another at each of the two cutoff values of $A-e/\delta$.

Proof of Proposition 3b In the Selling Equilibrium, p is given by (66), which is increasing in θ_{ID} and θ_{DI}. Given that in the Selling Equilibrium, θ_{ID} is decreasing in A but θ_{DI} is independent of A, p must be decreasing in A. That there is a discrete fall in p as the Selling Equilibrium turns into the Balanced Equilibrium can be established by showing that the denominator of (68), which gives p in the Balanced Equilibrium, is larger than that of (66) at any θ_{ID} and θ_{DI}. Moreover, by (68), p in the Balanced Equilibrium is also increasing in θ_{ID} and θ_{DI}, both of which are decreasing in A. Finally, that there is a discrete fall in p as the Balanced Equilibrium turns into the Buying Equilibrium can be established by noting that the numerator of (68) always stays strictly positive.

Proof of Proposition 3c By (33), (36), the restrictions in the first column of Table 2, and (75), in the Selling Equilibrium,

$$TV=\frac{e}{n^{D}}\left(n^{D}-n^{LD}+e\left(\frac{1}{\eta\left(\theta_{ID}\right)}-1\right)\right). \tag{94}$$

The result of the Proposition then follows, given that, by Proposition 3a, θ_{ID} is decreasing in A in the Selling Equilibrium. By (36) and (34) and the restrictions in the second column of Table 2, in the Balanced Equilibrium,

$$TV=\begin{cases} n^{LD}\eta\left(\theta_{ID}\right)\left(1-\mu(\theta_{DI})\right) & A\leq S+\frac{e}{\delta} \\ n^{LD}\mu\left(\theta_{DI}\right)\left(1-\eta(\theta_{ID})\right) & A>S+\frac{e}{\delta}\end{cases}. \tag{95}$$

The result of the Proposition then follows given that, by Proposition 3a, both θ_{ID} and θ_{ID} are decreasing in A in the Selling Equilibrium. By (32), (34), the restrictions in the third column of Table 2, and (79), in the Buying Equilibrium,

$$TV=\frac{e}{n^{D}}\left(n^{D}-n^{LD}+e\left(\frac{1}{\mu(\theta_{DI})}-1\right)\right). \tag{96}$$

The result of the Proposition then follows given that, by Proposition 3a, θ_{DI} is decreasing in A in the Buying Equilibrium.

Evaluate (94) and the first line of (95) at where $A=B_{M}+e/\delta$ and (79) yields the same value of $e\left(1-\frac{e}{n^{D}}\right)$. Evaluate the second line of (95) and (96) at where $A=B_{L}+e/\delta$ and (80) yields the same value of $e\left(1-\frac{e}{n^{D}}\right)$. This proves continuity.

Proof of Proposition 4a The condition $e \in \left[n^{LD}, n^{LD} + \frac{n^{SD}}{2}\right)$ is equivalent to $n^{LD} \in \left(2e - n^D, e\right]$. For such n^{LD}, the Balanced Equilibrium indeed holds if the condition in Proposition 2(b) is met, which can be rewritten as the first condition of the Proposition. Notice that the RHS of the condition is greater than $2e - n^D$ by Remark 1 in the proof of Proposition 2, meaning that any n^{LD} that satisfies the condition exceeds $2e - n^D$. Now when n^{LD} rises up to e, Proposition 2(a) applies. At $n^{LD} = e$, $B_L \to \infty$ and $B_M = B_S$, in which case the Balanced Equilibrium continues to hold. At $n^{LD} = n^D$,

$$B_M = S = B_L = \overline{B},$$

in which case the Balanced Equilibrium still holds only if $A - e/\delta = \overline{B}$. Otherwise, for $A - e/\delta < (>)\, \overline{B}$, the Selling (Buying) Equilibrium holds. In general, as n^{LD} increases from e to n^D, B_L falls from infinity, whereas B_M and S go up and diverge from B_S. Eventually the three bounds converge to $\overline{B}$. Then, for $A - e/\delta < (>)\, \overline{B}$, the Balanced Equilibrium must turn into the Selling (Buying) Equilibrium at some $n^{LD} \in \left(e, n^D\right)$. The cutoff values are from Proposition 2(a).

Proof of Proposition 4b In the Balanced Equilibrium, θ_{DI} is given by the solution to (89), whereas θ_{ID} can be recovered from (88) once θ_{DI} is known from the former equation. In the Buying Equilibrium, θ_{DI} and θ_{ID} are given by the solutions to (82) and (79), respectively. In the Selling Equilibrium, θ_{DI} is given by the solution to (75), whereas θ_{ID} can be recovered from (77) once θ_{DI} is known from the former equation. The comparative steady states followed straightforwardly from these equations. Just as in the proof of Proposition 3a, the continuity can be established by verifying that the equations for θ_{DI} and θ_{ID} for one equilibrium type coincide with another at each of the two cutoff values of $A - e/\delta$.

Proof of Proposition 4c In the Selling Equilibrium, p is given by (66), which does not directly depend on n^{LD}, given θ_{ID} and θ_{DI}. But then, the two market tightnesses in the Selling Equilibrium do not vary with n^{LD}. The proof for the jump in p that occurs when the Balanced Equilibrium gives way to the Buying or the Selling Equilibrium follow from Proposition 3b.

Proof of Proposition 4d In the Selling Equilibrium, TV, given by (94) is decreasing in n^{LD}, given that θ_{ID} is independent of n^{LD} in the Selling Equilibrium. In the Buying equilibrium, TV, given by (96) can be shown to be decreasing in n^{LD} with θ_{DI} given in (82).

In the Balanced Equilibrium, TV is given by either the first or the second line of (95). To show that both expressions are increasing in n^{LD}, we begin with noting that A^D, by (71) and (72), in the first instance is given by

$$A^D = A - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})},$$

But by (89),

$$A - \frac{e}{\delta} - \frac{e}{\eta(\theta_{DI})} = n^{LD} + n^D - \frac{e}{\mu(\theta_{DI})}$$

Because θ_{DI} increases with n^{LD} in the Balanced Equilibrium, the LHS strictly increases with n^{LD}, and so dealers hold more inventory in total. The RHS, however, can only rise by less than the increase in n^{LD}. Given that in the Balanced Equilibrium, $A^D = n_1^{SD} + n^{LD}$, a larger n^{LD} must be accompanied by a smaller n_1^{SD}. Also, according to (87), there would also have to be a smaller n_0^{SD}. In the investor-dealer market, both dealer-sellers and dealer-buyers execute e trades in the steady state; i.e.,

$$(n^{LD} + n_1^{SD})\eta(\theta_{ID}) = (n^{LD} + n_0^{SD})\mu(\theta_{DI}) = e$$

Because $n_1^{SD}\eta(\theta_{ID})$ and $n_0^{SD}\mu(\theta_{DI})$ strictly decrease with n^{LD}, $n^{LD}\eta(\theta_{ID})$ and $n^{LD}\mu(\theta_{DI})$ must be strictly increasing in n^{LD}. This implies that both the first and the second lines of (95) are strictly increasing in n^{LD}.

The proof of continuity is as in Proposition 3c.

Proof of Lemma 3 The objective function of the social planner is:

$$W = \max \left\{ \sum_{t=0}^{\infty} \beta^t n_H^{ON}(t)\, \upsilon \right\}, \tag{99}$$

with states $\left\{ n_H^{ON}(t), n_L^{ON}(t), n_B^I(t) \right\}$, initial conditions $\left\{ n_H^{ON}(0), n_L^{ON}(0), n_B^I(0) \right\} = \{\widehat{n}_H^{ON}, \widehat{n}_0^{ON}, \widehat{n}_B^I\}$, controls $\left\{ n_0^{SD}(t), n_1^{SD}(t), n_0^{LD}(t), n_1^{LD}(t), n_2^{LD}(t) \right\}$ and the following equations of motion:

$$n_H^{ON}(t+1) - n_H^{ON}(t) = -\delta n_H^{ON}(t) + n_B^I(t)\,\mu(\theta_{ID}[t]),$$

$$n_L^{ON}(t+1) - n_L^{ON}(t) = \delta n_H^{ON}(t) - n_L^{ON}(t)\,\eta(\theta_{DI}[t]),$$

$$n_B^I(t+1) - n_B^I(t) = e - n_B^I(t)\,\mu(\theta_{ID}[t]).$$

The constraints are given in (24)-(28) that hold at each moment in time, which can be summarized by the following two equations:

$$\theta_{ID}(t) = \frac{n_B^I(t)}{n^D - n_0^{SD}(t) - n_0^{LD}(t)},$$

$$\theta_{DI}(t) = \frac{n^D + n^{LD} - A - n_0^{LD}(t) + n_L^{ON}(t) + n_H^{ON}(t)}{n_L^{ON}(t)}.$$

In the above, a pair of $\left\{ n_0^{SD}(t), n_0^{LD}(t) \right\}$ uniquely determines the pair $\{\theta_{ID}(t), \theta_{DI}(t)\}$. This means that the controls in (99) can be stated in terms of the two market tightnesses only, whereby the admissible values are given by

$$\theta_{ID}(t) \in \left[\frac{n_B^I(t)}{\overline{n}_S^D(n_H^{ON}(t), n_L^{ON}(t))}, \frac{n_B^I(t)}{\underline{n}_S^D(n_H^{ON}(t), n_L^{ON}(t))} \right],$$

$$\theta_{DI}(t) \in \left[\frac{\underline{n}_B^D(n_H^{ON}(t), n_L^{ON}(t))}{n_L^{ON}(t)}, \frac{\overline{n}_B^D(n_H^{ON}(t), n_L^{ON}(t))}{n_L^{ON}(t)} \right],$$

with $\overline{n}_S^D$ and $\underline{n}_S^D$ denoting, respectively, the largest and smallest possible measures of dealer-sellers and $\overline{n}_B^D$ and $\underline{n}_B^D$ denoting, respectively, the largest and smallest possible measures of dealer-buyers, given state variables $n_H^{ON}(t)$ and $n_L^{ON}(t)$. Note that:
(1) To attain $\overline{n}_S^D$, first allocate one unit each of the assets to be held by dealers ($A - n_H^{ON}(t) - n_L^{ON}(t)$) to either small or large dealers, and then allocate one more unit each to large dealers if $A - n_H^{ON}(t) - n_L^{ON}(t) > n^D$.
(2) To attain $\underline{n}_S^D$, first allocate two units each of the assets to be held by dealers to large dealers, and then allocate one unit each to small dealers if $A - n_H^{ON}(t) - n_L^{ON}(t) > 2n^{LD}$.
(3) To attain $\overline{n}_B^D$, first allocate one unit each of the assets to be held by dealers to large dealers, and then allocate one unit each to either large or small dealers if $A - n_H^{ON}(t) - n_L^{ON}(t) > n^{LD}$.
(4) To attain $\underline{n}_B^D$, first allocate one unit each of the assets to be held by dealers to small dealers, and then allocate two units each to large dealers if $A - n_H^{ON}(t) - n_L^{ON}(t) > n^{SD}$.

To proceed, write (99) as

$$\begin{aligned} W\left(n_H^{ON}(t), n_L^{ON}(t), n_B^I(t)\right) = & \max_{\theta_{ID}(t),\theta_{DI}(t)} \left\{ n_H^{ON}(t)\,\upsilon_H \right. \\ & \left. + \beta W\left(n_H^{ON}(t+1), n_L^{ON}(t+1), n_B^I(t+1)\right)\right\}, \end{aligned}$$

in which the state variables for $t+1$ can be recovered from the equations of motions. There are four constraints corresponding to the four bounds of market tightness. Let $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ and $\lambda_4(t)$ be the respective Lagrange multipliers of the lower and upper bounds of $\theta_{ID}(t)$ and the lower and upper bounds of $\theta_{DI}(t)$.

Restricting attention to the steady state, we omit all time indices in the following. The first order conditions for $\theta_{ID}(t)$ and $\theta_{DI}(t)$ are then given by, respectively,

$$\beta n_B^I \mu'(\theta_{ID})(W_1 - W_3) + \lambda_1 - \lambda_2 = 0, \tag{107}$$

$$-\beta n_L^{ON} \eta'(\theta_{DI}) W_2 + \lambda_3 - \lambda_4 = 0. \tag{108}$$

In addition, there are three envelope conditions, one for each state variable:

$$\begin{aligned} W_1 = & \upsilon_H + \beta(1-\delta)W_1 + \beta\delta W_2 - \lambda_1 \frac{\partial(n_B^I/\overline{n}_S^D)}{\partial n_H^{ON}} + \lambda_2 \frac{\partial(n_B^I/\underline{n}_S^D)}{\partial n_H^{ON}} \\ & - \lambda_3 \frac{\partial(\underline{n}_B^D/n_L^{ON})}{\partial n_H^{ON}} + \lambda_4 \frac{\partial(\overline{n}_B^D/n_L^{ON})}{\partial n_H^{ON}} \end{aligned} \tag{109}$$

$$\begin{aligned} W_2 = & \beta(1-\eta(\theta_{DI}))W_2 - \lambda_1 \frac{\partial(n_B^I/\overline{n}_S^D)}{\partial n_L^{ON}} + \lambda_2 \frac{\partial(n_B^I/\underline{n}_S^D)}{\partial n_L^{ON}} \\ & - \lambda_3 \frac{\partial(\underline{n}_B^D/n_L^{ON})}{\partial n_L^{ON}} + \lambda_4 \frac{\partial(\overline{n}_B^D/n_L^{ON})}{\partial n_L^{ON}} \end{aligned} \tag{110}$$

$$W_3 = \beta\mu(\theta_{ID})(W_1 - W_3) - \frac{\lambda_1}{\overline{n}_S^D} + \frac{\lambda_2}{\underline{n}_S^D} \tag{111}$$

We first show that λ_3 must equal to 0. Suppose otherwise. By the definition of $\overline{n}_S^D$, $\underline{n}_S^D$, $\overline{n}_B^D$ and $\underline{n}_B^D$, λ_1, λ_2 and λ_4 must all equal to 0. Then equations (107) to (111) reduce to three equations, (108), (109) and (110) with only two unknowns, λ_3 and W_2. The set of λ_3 and W_2 that satisfy all three equations is of measure zero. Therefore, λ_3 must equal to 0.

By the same argument, we can show that λ_2 must equal to 0.

Next, we prove that $\lambda_1 > 0$. Suppose otherwise. Together with the fact that $\lambda_2 = 0$, this implies $W_1 = W_3 = 0$. We already know $\lambda_3 = 0$. Then equations (107) to (111) reduce to three equations, (108), (109) and (110) with only two unknowns, λ_4 and W_2. We have reached the desired contradiction.

Finally, we prove that $\lambda_4 > 0$. Suppose otherwise. Then by equation (108), $W_2 = 0$. Plug it into equation (110), we must have $\lambda_1 = 0$, which contradicts our previous conclusion that $\lambda_1 > 0$.

To summarize, we have shown that $\theta_{ID} = \frac{n_B^I}{\overline{n}_S^D(n_H^{ON}, n_L^{ON})}$ and $\theta_{DI} = \frac{\overline{n}_B^D(n_H^{ON}, n_L^{ON})}{n_L^{ON}}$. In other words, for efficiency, we should allocate the assets held by dealers to maximize the measure of dealers holding inventory and the measure of dealers having spare capacity: first allocate one unit each to large dealers; if $A - n_L^{ON} - n_H^{ON} > n^{LD}$, then allocate one unit each to small dealers; if $A - n_L^{ON} - n_H^{ON} > n^D$, then allocate one more unit each to large dealers.

Proof of Proposition 5 The allocations as described in Lemma 3 are the same as the allocations as described in the discussions following Proposition 2.

Proof of Lemma A1 The equation for p_{I_S} is from combining (21) and (60). The equations for p_{I_B} are from combining (23) and (65) for the Selling Equilibrium, (23), (60), and (67) for the Balanced Equilibrium, and (23), (70), and $p = 0$ for the Buying Equilibrium.

Proof of Propositions A1 and A2 By (46), given p, p_{I_S} depends only on and is increasing in θ_{DI}. In the Selling and Balanced Equilibria, $\partial\theta_{DI}/\partial A = 0$ and $\partial\theta_{DI}/\partial A < 0$, respectively. Then, $\partial p_{I_S}/\partial A$ has the same negative sign as $\partial p/\partial A$ in the two types of equilibrium. Next, in the Selling Equilibrium, $\partial\theta_{DI}/\partial n^{LD} = 0$, from which it follows that $\partial p_{I_S}/\partial n^{LD}$ has the same zero value as $\partial p/\partial n^{LD}$. In the Buying Equilibrium, by Lemma A1, $p_{I_S} = p = 0$. The discrete changes in p_{I_S} at where one equilibrium type changes to another follows from the discrete changes in p.

By combining (47) and (66), in the Selling Equilibrium,

$$p_{I_B} = \frac{\frac{1}{2}\beta\left(1-\beta+\beta\eta\left(\theta_{ID}\right)\right)\left(1-\beta+\frac{\eta(\theta_{DI})}{2}\beta\right)\upsilon}{\left(1-\beta+\left(\frac{\eta(\theta_{ID})}{2}+\frac{\mu(\theta_{ID})}{2}\right)\beta\right)\left(1-\beta+\frac{\eta(\theta_{DI})}{2}\beta\right)\left(1-\beta+\beta\delta\right)-\delta\beta^3\frac{\eta(\theta_{ID})}{2}\frac{\eta(\theta_{DI})}{2}},$$

where $\partial p_{I_B}/\partial\theta_{ID} > 0$. Then, given $\partial\theta_{ID}/\partial A < 0$, it follows that $\partial p_{I_B}/\partial A < 0$. Meanwhile, $\partial p_{I_B}/\partial n^{LD} = 0$ holds given $\partial\theta_{ID}/\partial n^{LD} = \partial\theta_{DI}/\partial n^{LD} = 0$ in the Selling Equilibrium. That p_{I_B} in the Buying Equilibrium, given by (49), does not vary with n^{LD} follows from $\partial\theta_{ID}/\partial n^{LD} = 0$ in said equilibrium. The discrete changes in p_{I_B} at which one equilibrium type changes to another can be verified by checking how, given θ_{ID} and θ_{DI}, p_{I_B} in (47) exceeds p_{I_B} in (48), which in turn exceeds p_{I_B} in (49).

Proof of Lemma A2 An S_1 can sell to an IB (investor-buyer), an L_0, or an L_1. He will not sell to an IB only if selling to other dealers yields a strictly larger surplus; i.e.,

$$\max\{z_{L_0,S_1}, z_{L_1,S_1}\} > z_{I,S_1}.$$

Expanding the expressions for the zs,

$$\max\left\{V_1^{LD} - V_0^{LD}, V_2^{LD} - V_1^{LD}\right\} > U_H^{ON} - U^B.$$

Subtracting $U_H^{ON} - U^{IB}$ from the two sides of the condition

$$\max\left\{V_1^{LD} - V_0^{LD} - \left(U_H^{ON} - U^B\right), V_2^{LD} - V_1^{LD} - \left(U_H^{ON} - U^B\right)\right\} > 0.$$

The two terms inside the max operator are simply the negatives of z_{I,L_1} and z_{I,L_2}, respectively. Then, the condition becomes

$$\max\{-z_{I,L_1}, -z_{I,L_2}\} > 0,$$

which is the same as

$$\min\{z_{I,L_1}, z_{I,L_2}\} < 0.$$

All this implies that if one type of dealer-seller finds it optimal not to sell to investor-buyers, then only one type of dealer-seller may find it optimal to do so. In any active steady-state equilibrium, indeed at least one type of dealer-seller must do so.

Now, suppose only S_1s sell to IB where

$$z_{I,S_1} = U_H^{ON} - U^B - \left(V_1^{SD} - V_0^{SD}\right) \geq 0. \tag{112}$$

An L_1 may then only sell to an S_0 or another L_1. Selling to an S_0 is optimal if

$$z_{S_0,L_1} = V_1^{SD} - V_0^{SD} - \left(V_1^{LD} - V_0^{LD}\right) \geq 0.$$

But if the condition holds,

$$z_{I,L_1} = U_H^{ON} - U^B - \left(V_1^{LD} - V_0^{LD}\right) \geq 0$$

must hold given (112). The hypothesis that only S_1 sell to IB then implies that selling to another L_1 must be optimal for the L_1 (otherwise the L_1 has no one to sell to), where

$$z_{L_1,L_1} = V_2^{LD} - V_1^{LD} - \left(V_1^{LD} - V_0^{LD}\right) \geq 0. \tag{113}$$

An L_2 may sell to an S_0 or an L_0 if selling to an IB is not optimal. Selling to an S_0 is optimal if

$$z_{S_0,L_2} = V_1^{SD} - V_0^{SD} - \left(V_2^{LD} - V_1^{LD}\right) \geq 0.$$

But if the condition holds,

$$z_{IB,L_2} = U_H^{ON} - U^B - \left(V_2^{LD} - V_1^{LD}\right) \geq 0$$

must hold given (112). The hypothesis that only S_1 sell to IB then implies that selling to an L_0 must be optimal for the L_2, where

$$z_{L_0,L_2} = V_1^{LD} - V_0^{LD} - \left(V_2^{LD} - V_1^{LD}\right) \geq 0. \tag{114}$$

The two conditions, (113) and (114), together imply that

$$V_1^{LD} - V_0^{LD} = V_2^{LD} - V_1^{LD}.$$

Thus, if neither L_1s nor L_2s find it optimal to sell to investor-buyers or to small dealers, large dealers do not gain by selling and buying among themselves either. They must then be inactive in equilibrium.

Next, suppose only L_1s sell to investor-buyers, where

$$z_{I,L_1} = U_H^{ON} - U^B - \left(V_1^{LD} - V_0^{LD}\right) \geq 0. \tag{115}$$

An S_1 may sell to an L_0 or to an L_1 if not selling to an investor-buyer. If the first sale is optimal, it must be optimal for the S_1 to sell to an IB as well given (115). The hypothesis that only L_1 sells to investor-buyers then requires that it is optimal for an S_1 to sell to an L_1 where

$$z_{L_1,S_1} = V_2^{LD} - V_1^{LD} - \left(V_1^{SD} - V_0^{SD}\right) \geq 0. \tag{116}$$

An L_2 may sell to an L_0 or to an S_0. If the first sale is optimal, it must be optimal for the L_2 to sell to an IB as well given (115). The condition for the second sale to be optimal is that

$$z_{S_0,L_2} = V_1^{SD} - V_0^{SD} - \left(V_2^{LD} - V_1^{LD}\right) \geq 0. \tag{117}$$

The two conditions, (116) and (117), together imply that

$$V_1^{SD} - V_0^{SD} = V_2^{LD} - V_1^{LD}.$$

Thus, if neither S_1s nor L_2s find it optimal to sell to investor-buyers, S_1s only sell to L_1s, where such trades do not yield any surplus. This implies that small dealers must be inactive in equilibrium.

The case for where only L_2s sell to investor-buyers can be shown in a similar way to imply that small dealers must be inactive in equilibrium.

The proof that in any equilibrium in which both small and large dealers are active, investor-sellers must sell to all three types of dealer-buyers can be constructed similarly.

Proof of Proposition A3 Substituting in the prices, we can rewrite dealers' value functions as follows.

$$rV_0^{SD} = \mu\left(\theta_{DI}\right)\frac{z_{S_0,I}}{2} + \alpha\left\{\frac{n_1^{LD}}{2n^D}\max\{z_{S_0,L_1},0\} + \frac{n_2^{LD}}{2n^D}\max\{z_{S_0,L_2},0\}\right\},$$

$$rV_1^{SD} = \eta\left(\theta_{ID}\right)\frac{z_{I,S_1}}{2} + \alpha\left\{\frac{n_0^{LD}}{2n^D}\max\{z_{L_0,S_1},0\} + \frac{n_1^{LD}}{2n^D}\max\{z_{L_1,S_1},0\}\right\},$$

$$rV_0^{LD} = \mu\left(\theta_{DI}\right)\frac{z_{L_0,I}}{2} + \alpha\left\{\frac{n_1^{SD}}{2n^D}\max\{z_{L_0,S_1},0\} + \frac{n_2^{LD}}{2n^D}\max\{z_{L_0,L_2},0\}\right\},$$

$$rV_1^{LD} = \mu(\theta_{DI})\frac{z_{L_1,I}}{2} + \eta(\theta_{ID})\frac{z_{I,L_1}}{2} +$$
$$\alpha\left\{\frac{n_0^{SD}}{2n^D}\max\{z_{S_0,L_1},0\} + \frac{n_1^{SD}}{2n^D}\max\{z_{L_1,S_1},0\} + \frac{n_1^{LD}}{2n^D}\max\{z_{L_1,L_1},0\}\right\},$$

$$rV_2^{LD} = \eta(\theta_{ID})\frac{z_{I,L_2}}{2} + \alpha\left\{\frac{n_0^{SD}}{2n^D}\max\{z_{S_0,L_2},0\} + \frac{n_0^{LD}}{2n^D}\max\{z_{L_0,L_2},0\}\right\}.$$

Suppose $V_1^{SD} - V_0^{SD} > V_1^{LD} - V_0^{LD}$. Then $z_{I,S_1} < z_{I,L_1}$ and $z_{S_0,I} > z_{L_0,I}$. Together with the fact that $z_{L_1,I} \geq 0$, this implies that

$$\eta(\theta_{ID})\frac{z_{I,S_1}}{2} - \mu(\theta_{DI})\frac{z_{S_0,I}}{2} < \mu(\theta_{DI})\frac{z_{L_1,I}}{2} + \eta(\theta_{ID})\frac{z_{I,L_1}}{2} - \mu(\theta_{DI})\frac{z_{L_0,I}}{2}.$$

Also, $V_1^{SD} - V_0^{SD} > V_1^{LD} - V_0^{LD}$ implies that $z_{S_0,L_1} > 0 > z_{L_0,S_1}$, $z_{S_0,L_2} > z_{L_0,L_2}$, and $z_{L_1,S_1} < z_{L_1,L_1}$. This means

$$\frac{n_1^{LD}}{2n^D}\max\{z_{S_0,L_1},0\} + \frac{n_2^{LD}}{2n^D}\max\{z_{S_0,L_2},0\} > \frac{n_1^{SD}}{2n^D}\max\{z_{L_0,S_1},0\} + \frac{n_2^{LD}}{2n^D}\max\{z_{L_0,L_2},0\}$$

and

$$\frac{n_0^{LD}}{2n^D}\max\{z_{L_0,S_1},0\} + \frac{n_1^{LD}}{2n^D}\max\{z_{L_1,S_1},0\}$$
$$< \frac{n_0^{SD}}{2n^D}\max\{z_{S_0,L_1},0\} + \frac{n_1^{SD}}{2n^D}\max\{z_{L_1,S_1},0\} + \frac{n_1^{LD}}{2n^D}\max\{z_{L_1,L_1},0\}$$

The above three inequalities together imply that $V_1^{SD} - V_0^{SD} < V_1^{LD} - V_0^{LD}$. This is a contradiction.

Now suppose $V_2^{LD} - V_1^{LD} > V_1^{SD} - V_0^{SD}$. Similarly, we can show that this implies $z_{L_1,I} > z_{S_0,I}$, $z_{I,L_2} < z_{I,S_1}$, $z_{S_0,L_2} < 0 < z_{L_1,S_1}$ and $z_{L_0,L_2} < z_{L_0,S_1}$. These inequalities in turn imply that $V_2^{LD} - V_1^{LD} < V_1^{SD} - V_0^{SD}$. This is a contradiction.

Given that we have shown $V_1^{LD} - V_0^{LD} \geq V_1^{SD} - V_0^{SD} \geq V_2^{LD} - V_1^{LD}$, it is straightforward to verify that the two equalities hold are strict unless $z_{I,L_1} = 0$.

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